THE TEACHING OF ALGEBRA IN SCHOOLS

A REPORT

PREPARED FOR THE

MATHEMATICAL ASSOCIATION

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* Apart from § 1 and § 17 the topics are arranged alphabetically. The Appendices contain a fuller discussion of directed number and some miscellaneous teaching hints.

APPENDICES

The Teaching of Algebra in Schools

§ 1. INTRODUCTION

§ 1·1. REASONS FOR, AND AIMS IN THE TEACHING OF ALGEBRA

It is never easy to make a convincing case for the teaching of any subject—for its teaching, that is, under ordinary school conditions to all and sundry. Long established tradition and the eloquence of some generations of schoolmasters failed to convince the world that the classical languages must be learnt by all who aspired to be thought educated or to rise above the most lowly stations in life. Nor have specialists and enthusiasts for other subjects been more successful—unless indeed the congestion of curriculum under which we suffer, and of which we complain, may be counted success.

That mathematics is a necessity for serious students of science and for many who are concerned with the applications of science, whether physical, biological or economic, is of course obvious ; but these, though numerous, form but a relatively small fraction of those who pass through our schools ; and the questions, why and how should we teach mathematics, concern the larger residue who will never go far and will probably never make any direct use of the little they have time to learn. Custom at present gives the subject a place in the school course, and perhaps the main point for those who think the custom should be maintained is so to do their work that their pupils will not be provoked to ask 'Why should I be bothered to learn this stuff ? ' But this involves for every teacher much consideration of the question 'how ? ' and at least such consideration of the question 'why ? ' as will guide him in the choice of what he shall do, seeing that he certainly cannot do everything in the limited time at his disposal.

It is therefore relevant to consider what parts or aspects of elementary mathematics—or in the present connection, of elementary algebra—should be stressed, and what service they may be expected to render to general intellectual and moral development.

Historically, algebra grew out of arithmetic, and it ought so to grow afresh for each individual. Arithmetic generally involves the combination of two different processes of thought : (*a*) analysing a situation so as to see what ought to be done, (*b*) carrying out the

operations so decided upon. (In mere sums set for exercise the process (a) is often suppressed and the child simply told to do this or that piece of mechanical addition or multiplication.)

The fundamental use of algebra seems to attach to (a) : it affords a compact symbolism in which we record what we decide must be done ; and by this very fact it compels, or at least conduces towards an accurate analysis. But more than that—a point of the highest importance—the symbolisation involves, or brings to light, generalisation ; it helps us to realise the wide applicability of a single statement or that a multitude of individual facts are included in one general rule.

Even if this were the whole story—if algebra were merely a convenient shorthand for the recording of rules and a means of generalisation—it would be of value in itself and its acquisition would involve a very valuable mental discipline ; but there is much more.

An algebraic statement, once made, often proves to be capable of transformation into other forms which greatly simplify the calculations required ; the solution of problems by equations and the reduction of labour by judicious factorisation are examples of this. Algebra, that is, becomes a machine for facilitating both bare calculation and the solution of complicated problems in all regions of thought in which number or measurement plays a part. Further—a very curious point—the machine proves to be creative. It presents us in its operation with queer results or incidental features—such as negative and 'imaginary' numbers—which suggest new ideas and fresh applications.

In yet another, and perhaps even more immediately important, way is it creative : an expression being applicable to a wide range of values of the letters involved invites consideration of the way in which its value changes as one or other of its component letters varies—and we are led to the idea of functionality.

As we have already noticed, the application of arithmetic and also algebra to any question or field of thought to which it is applicable normally involves two distinct processes : (a) analysing the situation and expressing in words or in symbols the calculations which will be necessary ; (b) performing those calculations for the particular values of the numbers involved. Between these there may and generally will intervene : (c) such transformation of the original expression (whether in words or in symbols) as will bring most clearly into prominence the result required or will reduce as far as possible the labour of effecting the actual calculations.

Of these three, (a), (b) and (c), clearly the most fundamental is (a) : this always involves real thought and realisation of the facts—from the simplest problem presented to a child in which he has merely to decide whether to add, subtract, multiply or divide, to the most abstruse of scientific questions ; (b) is mere direct numerical cal-

culation; (c) represents the mechanical or manipulative side of the subject.

It follows that, while in practice (b) is indispensable and (c) very useful, neither is of any use at all without (a). Hence the fundamental necessity in the teaching of algebra is to give training in analysis and expression.

It does not necessarily follow that this, though the most important, is also the first thing to be done, and until comparatively recent years the general practice was to start with (c), i.e. with the manipulation of algebraic symbols. Thus, in a well-known popular book of the later years of the last century, the order of chapters is: I, Definitions, Substitutions; II, Negative Quantities, Addition of like terms; III, Simple Brackets, Addition; IV, Subtraction; V, Multiplication; VI, Division; VII, Removal and Insertion of Brackets; VIII, Simple Equations; IX, Symbolic Expression; X, Problems leading to Simple Equations.

Thus it was not till Chapter IX—if, as usually happened, the teacher followed the order of the book—that the essential element in the subject was reached. When it was reached, it was often found difficult and, if it was not actually scamped, failure was cloaked by facility in formal manipulation which meant nothing to the manipulators: the ordinary examination paper would have a question representing each chapter, and if those on the first eight chapters could be answered, what did failure on IX and X matter? It was only what was expected, just as it sufficed in 'Euclid' to know definitions and propositions, total inability to do 'riders' being taken for granted.

Of course such failure was no necessary outcome of the method, and with the cleverer or more mathematically inclined boys it did not occur. None the less it has come to be generally recognised that this arrangement, quite in place in the orderly systematic exposition of a treatise, is not the natural order for teaching children and therefore is not the natural order for a first schoolbook.

The change of view is identical with that which has occurred in regard to most subjects, languages, music, drawing, etc.; generally it is felt that a more immediate approach to the essential stuff of the subject is desirable, as contrasted with a long preliminar training in the mechanical elements.

As in all such changes of fashion, revolt may have gone too far, and the value attached to mechanical skill may have been unduly depreciated; but there can be little doubt that the change has greatly increased the interest of the early stages of the subject. It has also increased the difficulty of those stages, but only because it has brought the essential element of the subject to the forefront, whereas formerly it was possible to shirk it.

§ 1·2. THE 'OUTLOOK' VALUE OF ALGEBRA

The foregoing section has shown that the stress in teaching elementary algebra should be on the analysis of a problem that will yield to mathematical treatment and the expression of an argument based on this preliminary analysis, and not on the acquisition of manipulative skill. But it may still be said by opponents of the subject that mathematical teachers have only climbed out of the morass of formal training on one side to fall into it on another. Can such powers of analysis and expression be transferred from algebra to habits of thought in real life ? And could they not be equally well acquired by learning to translate English into Latin prose, or even by learning to solve problems in bridge or chess ?

The question that remains to be answered then is, 'Why learn algebra to acquire this power of analysis, rather than Latin prose or chess ? ' The answer can only be that algebra gives something more than mental training and that it has a definite contribution to make to cultural education.

The average boy is not going to use algebra in after life, so that the utilitarian argument can make little appeal to him except in so far as he can see that other people may find a knowledge of algebra a 'useful' acquisition. But if he is to become an educated and cultured man (and that is presumably the object of a general as opposed to a technical education), he should learn something of the part that mathematics has played and continues to play in the development of the modern world. He may never get as far as studying the elements of mechanics and the calculus, but he can at least be made to realise that without some means of making general statements it is impossible to arrive at general truths or to apply mathematics to the problems of the engineer, the scientist, or the economist. Hence the importance of stressing at all stages the idea that letters in algebra may apply to *any* number—not necessarily to some particular unknown number which will solve some particular problem, but to any one of a large class of numbers to which a statement might apply ; in fact, the importance of stressing 'x' as representing an independent variable, and not only as an unknown.

The joys of pure mathematics, the study of algebraic form, the appreciation of an 'elegant' method are for the pure mathematician and not for the average boy. Not that the average boy may not be expected and encouraged to derive real intellectual pleasure from his study of the subject. It would be an arid subject if there was no joy of achievement to be experienced. Many boys have an interest in number as such and appreciate such a general statement as 'the sum of the first n odd numbers is n^2' ; many will appreciate the joy of discovering an unknown number, especially when they are able to test for themselves, independent of the teacher's authority

or the answers at the end of the book, that they really have found a number which satisfies the conditions of the problem; some, but comparatively few, will appreciate the beauty of the algebraic form a^2-b^2 or the ingenuity of the proof of the Remainder Theorem. These few will look after themselves, and it is the others who will require encouragement if they are not to wonder at intervals, 'Why am I learning this stuff?'

They must, then, be taught to appreciate the outlook value of the subject; to see that, without it, generalisation of scientific truths into simple concise formulae is impossible; that it is the road which leads to scientific discovery for others, and for themselves to an understanding of the results that the pioneers have achieved. If they are to be guided successfully along this road, they must continually be shown to what kind of goal it is leading; they must be shown the value of general statements about any number or set of numbers; they must be given an idea of x as an independent variable, and the idea of functionality wherever the value of a 'y' is found to depend upon the value of an 'x'. Above all the teacher must get out of his head the opinion that juggling with x and y has any special merit of its own which cannot equally well be found in other formal subjects of the curriculum. In guiding his pupils along their road he must not allow them to struggle unnecessarily in the slough of algebraic manipulation. Many will eventually be able to climb the Delectable Mountains of the Calculus; and all should see that the climb can be made, and should anyhow reach a point on the path from which they can see the vista beyond, with some glimpses, however blurred and distant, of the Promised Land.

§ 1·3. SOME FURTHER REMARKS ON TEACHING ALGEBRA

There is not, or ought not to be, any such isolated subject as elementary algebra in the curriculum. There is a subject, elementary mathematics, and it is inevitable that algebra should be part of it. This fusion of elementary mathematics is one of the fundamental changes that has taken place in mathematical teaching during the present century.

A letter in algebra may represent *any* number whatever, or any number belonging to some large class, e.g. the class of positive integers. The all-important idea of '*any*' (see A. N. Whitehead, *Introduction to Mathematics*) can scarcely hope to thrive in an atmosphere other than an algebraic one.

It is worse than useless to introduce to the pupil, all at once, an immense body of symbols for which he sees no use. But he will already have met with operational symbols ($+$, $-$, $\times$, $\div$) in arith-

metic, and there are other signs ($\therefore$, $=$, $>$, $<$) the economy of which can be made plain to him.

Of the formal laws of algebra the only one which it may be useful to mention in the classroom is the law

$$p(a+b) \equiv pa+pb.$$

This acts as a bond between statements not obviously connected. As instances we would take

$$5(a+b)=5a+5b,$$
$$\tfrac{1}{2}(a+b)=\tfrac{1}{2}a+\tfrac{1}{2}b,$$
$$-(a+b)=-a-b,$$
$$(x+y)(a+b)=(x+y)\,a+(x+y)\,b,$$
$$(x+y)(a+b)=x(a+b)+y(a+b),$$

and it should be pointed out when occasion arises that the symbols $\sqrt{}$, log and sin are not distributive over addition, i.e. that $\sqrt{(a+b)}$ is not equal to $\sqrt{a}+\sqrt{b}$, or $\log(a+b)$ to $\log a+\log b$ or $\sin(a+b)$ to $\sin a+\sin b$. Whether the word 'distributive' is mentioned explicitly or not may be left to the teacher.

Drill in mere manipulation is necessary at every stage in school algebra. That this should be thorough, so far as it goes, will be admitted by all teachers, but it should in the main be given *after* its necessity in applications has been perceived by the pupil and not *before*; also, it should not be carried further than is needed to ensure facility in these applications.

Success or failure may in many mathematical tasks be made plain to the pupil as a matter beyond dispute and independently of the authority and taste of the teacher; this is especially true of the tasks imposed in algebra.

Checking the result is a much more normal, simple, and satisfactory process in algebra than it is in arithmetic, because it is never merely a matter of going over the same process again more carefully. Checking simple equations should not look like solving simple equations. Factors should be checked by multiplication. Quadratic equations may be checked by the sum and product of the roots. Other checks come from ideas of degree or form or (later) from such things as sums of suffixes; so that, speaking generally, algebra is pre-eminent as the subject in which checks are most satisfactorily and constantly applied.

Algebra, well-taught, should develop in the pupil the power of expression in the English language—helped out it is true by appropriate symbols, but rendered more exact, not more slovenly, by their use.

Can the impartiality acquired in learning algebra be transferred to human problems? A fault of many thinkers is to attack a problem with the intention of arriving at this solution rather than that. To acquire the spirit of the mathematician, whose concern is to find the true solution, would go far to checking this pernicious habit.

§ 2. BEGINNING ALGEBRA *

§ 2·1. GENERAL DISCUSSION OF THE METHODS OF APPROACH

There are three distinct methods of introducing the subject of algebra; by means of 'The Four Rules', 'Problems' and 'Formulae'. School books are in use which approach algebra from one or other of these starting-points, more or less to the exclusion of the other two. This appears to be undesirable, it being preferable to blend the last two methods.

The 'Four Rules' method is unsatisfactory, not because it is difficult, but because it is uninteresting and leaves the boy in a very bad position for further progress. He will have learnt to regard the new subject as meaningless and artificial, and this is the most difficult of all defects to remedy. A certain amount of teaching of the four rules is inevitable in the early stages. They familiarise the boy with the alphabet and grammar of the new language: the ground is to some extent familiar as an extension of arithmetic; and he must gradually acquire manipulative skill if he is to make any progress. But he will have no desire to make progress if he does not learn what the subject is about and for what purposes manipulative skill must be acquired.

In both the 'Problem' and the 'Formula' methods these rules of manipulation do not appear in an arbitrary fashion, but they are developed as the boy feels the need for them. An appeal is made to his own reason and not to arbitrary authority. It is not easy to decide whether problems or formulae are likely to make this appeal more persuasively or convincingly, and teachers are divided on the subject.

The early introduction of problems has many advantages. It is in accordance with the historical development of the subject, and it awakens a boy's interest by appealing to the puzzle-solving instinct. Also a boy who has begun algebra by work on problems will not be likely to find so much difficulty in solving them as many boys find who have not met them till a later stage. But there is a danger that the subject will soon appear to be trivial, if there is an excessive preoccupation with the solution of equations. In any event the

* The reader should note that henceforward the sequence of topics is alphabetical.

early problems should not be restricted to those which happen to lead to simple equations ; a wide range of topics should be introduced, whether the resulting equations can be solved or not. And it is necessary to condemn unhesitatingly a common 'cram' method of teaching algebra, by which nothing is taught but the solution of simple, simultaneous and quadratic equations with a few standard factors to be learnt by heart. This may help a boy to satisfy an examiner, but it will not educate him.

In the problem method of introducing algebra some formula work occurs immediately in connection with problems, but it arises only incidentally, so that there is danger that the boy may learn at first to regard the idea of x as representing an unknown in an equation as more important than the idea of x as a variable. In the formula method the idea of x as a variable is in the forefront from the start.

If the general principle is accepted that the first thing to attend to is the symbolical expression of processes to be performed, there is not really any essential difference between the two methods. It is largely a matter of initial emphasis. The advantage of introducing formulae early is that they give a clear idea of the main trend of the subject, [illegible] the statement in symbolic language of general truths and the discovery of fresh truths with the assistance that this symbolic language gives in economising thought. Also they serve to bring the subject into close relation with its early practical applications, and interest can thus be aroused. The formula method is more exacting for the teacher and more difficult for the slower boy. Many boys find difficulty in generalising the simplest statements, and each one of a set of examples may be a real difficulty to some boy or other in an ordinary class. In this method there is less obvious opportunity and just as much necessity for familiarising the class with new processes by steady drill and so developing the necessary manipulative skill. In fact, there are fewer guiding lines for teacher and pupil to follow, but it can be maintained that the ultimate goal to be reached comes more readily into view if the start is made on these lines.

It may be said definitely that it is a mistake to lay emphasis on either problems or formulae to the exclusion of the other. A more detailed discussion of a way in which the combination of the two methods can be made is given in the next section.

§ 2·2. SOME DETAILS CONCERNING THESE METHODS

(·21) In the course of their arithmetic boys should have been introduced to the use of letters to represent numbers. If this has not been done, it is useful, before beginning algebra, to revise arith-

metic and perhaps some geometry, introducing letters in questions such as these:

(a) How many pence are there in one shilling? in two shillings? in S shillings?

(b) What is p % of £5?

(c) Give the number which is 6 less than x.

(d) The breadth of a rectangle is 3″ and the length is x″ longer than the breadth. What is the perimeter of the rectangle?

This work serves three purposes:

(i) It makes the class familiar with the use of letters to represent numbers.

(ii) It helps them to pick out the essential features of the arithmetical processes with which they are familiar.

(iii) It provides a fresh drill-ground for the fundamentals of arithmetic.

(·22) Once this work has been done, a class is ready for a first, informal, algebra lesson. It is a good thing if a number of simple general rules are known, e.g. those for finding areas, volumes, angle-sum of a polygon.

The chief difficulty in using this method is the lack of sufficient knowledge of such general rules.

The purpose of the first lesson will be the condensation of simple rules into formulae, using the 'shorthand of algebra', i.e. choosing suitable symbols and arranging them according to certain conventions. It is best to begin by using letters that suggest the words, e.g. A for area, P for perimeter, l for length, etc. Care should be taken to make the formulae that are used very simple at first, e.g. area or perimeter of rectangle, circumference of circle, etc., those constructed by the pupil himself being especially valuable. Most of the quantities introduced should be familiar to the boy, but others should also be used to suggest the wider field to which elementary mathematics can be applied; e.g. a boy is sure to be interested by examples on the R.A.C. formula for the H.P. rating of a motor car.

After constructing formulae the next step is to use them.

Their natural use is 'forwards', e.g. if $S=2n-4$ gives the sum of the angles of a polygon, find S for known values of n. Here appropriate selection of examples will enable any part of the usual work on substitutions and evaluations to be covered. Such work, however, should not be carried far at this stage nor should the arithmetic be made laborious. If it is desired to give practice in fractions and decimals by evaluating formulae, this should be regarded as arithmetical drill.

At an early lesson the next, perhaps unexpected, advance should be made. It should be pointed out that some formulae may be

worked 'backwards', e.g. if in $S=2n-4$ it is given that $S=20$, we may find n. Again, given the formula $P=2(l+b)$ for the perimeter of a rectangle, we may find b for given numerical values of P and l. Here we encounter the simple equation, or, if we start with the construction of the formula, the 'problem'. It is natural also to show the class how the equation arises in other ways, e.g. in the 'think-of-a-number' type of problem.

(·23) Alternatively, algebra proper may be begun with the problems of the 'hidden number' type. Even a small boy can find the value of x in the equation $5+x=12$ without having first to learn the hard and fast rules for the solution of simple equations. All he need ask himself is 'what number added to 5 gives 12'? The question is one with which he is very familiar from the arithmetic lesson.

This type of problem leads naturally to simple equations, and the rules for dealing with equations can then be developed by *viva voce* teaching. After this the work consists of symbolical expression, problems and equations. The symbolical expression will be such as is needed for the problems that will be done. It must have a set purpose and not be too complicated. The problems must be chosen so that they give rise to the desired types of equations. Boys may at times invent their own 'find-the-number' problems, but this work must be carefully supervised by the teacher. The problems and equations must be chosen so that they introduce the various pieces of manipulation which the class has to master.

If this alternative start has been adopted it will now be time to turn to the formula and to proceed with the work thereon set out in paragraph (·22) above.

(·24) We have now a variety of aspects of the new subject, which can be discussed in short periods of oral as well as in written work. There is the construction of the formula or symbolical expression, its use for direct calculation, and the methods for dealing with equations which are obtained by leaving one of the symbols of the formula as an unknown.

Attention should also be given to the retranslation into words of a formula given in algebraic notation.

A certain amount of formal drill will be necessary, e.g. on brackets, on the index notation and on collecting terms.

(·25) *Some suggestions as to separate points.*

Substitutions. By careful choice of examples it is possible to deal with one difficulty at a time, e.g. the following formulae each bring in some fresh point:

(*a*) $A=4\pi r^2$,	(*d*) $A=\pi(r_1+r_2)(r_1-r_2)$,
(*b*) $V=\frac{4}{3}\pi r^3$,	(*e*) $C=\frac{5}{9}(F-32)$,
(*c*) $S=2\pi r(r+h)$,	(*f*) $T=2\pi\sqrt{(l/g)}$.

Indices. Easy practice in drill work involving indices is given by questions involving areas and volumes, e.g. the edges of a brick are a, $2a$, $3a$ units, find its surface and volume, or, find the area of wood needed for a match-box and its cover if its length, breadth and depth are $7x/2$, $2x$, x units.

Extensions. Boys will often extend the range of classroom algebra by bringing forward formulae they have seen in diaries or technical journals. Any particular difficulty of symbolism occurring in these will be worth discussing.

§ 3. DIMENSIONS AND DEGREE

Most boys when they come to their first geometry lesson have already acquired from their experience of things around them a general perception of geometrical shape. But the form of an algebraical expression, the pattern in which it is cast, presents to them a physiognomy that remains for some time unfamiliar and perplexing.

Moreover the same expression may assume different masks:

$a^2 - a - b^2 + b$ is a four-term expression, a quadrinomial function of a and b;

written as $a^2 - a - b(b-1)$ it appears as a trinomial quadratic function of a;

as $(a^2 - a) - (b^2 - b)$ it is the difference between two terms, one a function of a, the other the same function of b;

as $(a^2 - b^2) - (a - b)$ it is again the difference between two terms, the second being a 1st degree function of a and b, the first being the corresponding 2nd degree function.

This protean quality of the algebraical expression naturally increases the pupil's perplexity. It is part of his training in algebra to give him a clear vision to distinguish these different aspects of the same expression, and to realise that expressions that look different may really be the same.

The teacher may use freely the words 'degree' and 'function' long before he expects the pupil to use them rightly himself. The teacher should use them in such a context and with such explanatory periphrasis as will leave no ambiguity in the boy's mind.

Simple and simultaneous equations may be referred to as equations of the 1st degree in one and two unknowns; quadratics as of the 2nd degree.

The idea of 'degree' is inherent in the measures of length, area and volume. Reference to these concepts and to the representation of linear measurements by single letters will lead to the correspondences:

linear measure,	a, $a+b$,	1st degree;
square "	a^2, xy, x^2-y^2,	2nd "
cubic "	a^3, a^2b, abc,	3rd "

A review of such formulae as the pupil knows for the circumference of a circle, the perimeter of a room, the areas of rectangle,

trapezium, circle, the volumes of cuboid, sphere, cylinder will impart reality to the idea.

In multiplication and division the pupil gets drill in recognising the number and degree of the terms and learns the advantage of arranging expressions in ascending or descending order of powers. He should have questions to do in which this point is driven home. He perceives (i) that the degree of the leading term of a product is the sum of the degrees of the leading terms of the factors, (ii) that when homogeneous expressions are multiplied, the product is homogeneous, and (iii) corresponding facts for division. Reference to mensuration gives concrete confirmation, e.g. if a cistern of volume V cu. ft. has a base of area ab sq. ft., the depth d ft. is given by $d=V\div(ab)$.

With some such start he tackles factorisation better equipped to overcome its difficulties. Understanding the pattern of an expression as presented to him, he has less difficulty in recasting it into one of the few standard elementary patterns—really only two, $pa+pb$ and ax^2+bx+c.

He also knows (i) that factorisation cannot affect the degree any more than area can be expressed in any but square measure and (ii) that homogeneous expressions will have homogeneous factors: thus

$$\begin{aligned} a^2-ab-d^2+bd &= a(a-b)-d(d-b) \\ &= (a-d)(a-b)(d-b) \end{aligned}$$

is palpably wrong though very common;

$$a^3-b^3 \text{ cannot be } (a^2-b)(a+b^2),$$

it must factorise as $(a \dots b)(a^2 \dots b^2)$.

On the other hand, when he perceives that the factors conform in degree to the character of the expression he has an indication (but not a certainty) that his result is correct.

Trained on these lines he acquires a sense of control and has at his disposal one of the best of checks. He can apply the same checks to the simplification of fractions and other manipulative work, e.g. $\frac{x}{y}+\frac{y}{x}$ cannot be x^2+y^2.

It will come as no surprise to him, it may even be an expectation that graphs of the functions which conform to a certain pattern will have a corresponding definite geometrical shape, e.g. the graphs of ax^2+bx+c for all values of a, b, c are parabolas, and that a straight line cannot cut such a graph in more than two points. If the graph he draws for a 2nd degree function can be cut by some line in three points, he has made a mistake in substitution or plotting.

He will recognise that every trigonometrical formula must be homogeneous. Thus if he has proved $a/\sin A=b/\sin B=c/\sin C$,

by drawing a perpendicular he knows that each expression is of the 1st degree and represents the length of some line, and he will be interested in finding or proving that it is the diameter of the circum-circle.

In Mechanics the question of dimensions * assumes considerably greater importance, and the boy well trained in Algebra will be alive to the significance of the dimensions in the formulae he meets in the subject.

* See *Report on the Teaching of Mechanics*, § 7·05.

§ 4. EQUATIONS

Historically algebra has been much concerned with methods of solving equations. It is a long step from the methods of Diophantus to those of Cardan and Ferrari. Equations which Diophantus rejected as 'impossible' were rendered 'possible' when negative numbers, irrational numbers and finally complex numbers were introduced. Hence eventually the general theorem that an equation of the nth degree has n roots could be stated and proved, one of the most striking examples of the stimulating generality at which Mathematics always aims by merging particular instances in a general law.

It is easily possible to spend too much time with beginners on equations. A warning has already been given against this tendency. Equations should, with beginners, arise either from problems or from the use of formulae when one of the letters in the formula is unknown. The object of this method of introducing them is to attract attention in the first place to their uses. Subsequently there must be steady drill in methods of solving equations; the weakest pupils derive some benefit from this drill because they find something that they can do successfully and they acquire a sense of mastery, however feeble. Their natural desire to know the use of it all must at times be satisfied, but for considerable periods it may be in abeyance whilst they enjoy the sensation of acquiring a technique.

There follows a detailed elementary discussion of various points in the teaching of equations, some of which has already appeared in the *Report on the Teaching of Mathematics in Preparatory Schools*

§ 4·1. SOME POINTS IN THE TEACHING OF EQUATIONS

The methods used in solving equations should be based on first principles (the four axioms): the idea of a balance should be kept before the pupils *; so long as the same thing is done to both sides the balance will be maintained. The things which can be done are the four simple operations of arithmetic. But it must not be assumed that the 'four axioms' are already in the pupil's mind ready for reference; they must be arrived at as generalisations of very easy numerical cases in which the result is arrived at intuitively.

E.g. for $x+5=7$, the beginner will see that $x=2$, since $2+5=7$. How is 2 obtained from the 7 and 5? As $7-5$. (Give several such.)

* In fact the original meaning of the word 'equation' is 'keeping level' or 'making level'.

What process has been adopted? Subtracting 5 from both sides. Similarly, if $x+423=564$, then $x=564-423$, subtracting 423 from each side. This process of subtracting a number from each side may be used always (other examples follow). It may be generalised in the form, if $x+a=b$, then $x=b-a$. The same sort of process should be used in teaching that addition to, multiplication of, or division of each side by any number is legitimate.

The use of the symbol = needs care. It is best to have only one = in each line and to arrange that = signs in successive lines shall be put one under another.

The solution of an equation should read as a consecutive series of sentences each containing a verb (viz. the sign =) connected by conjunctions (i.e. the sign ∴). The common error of replacing the ∴ by = should be carefully avoided since it involves giving two different meanings to the sign =. The correct use of this sign is to assert that two numbers are equal; the misuse asserts that one statement is logically equivalent to the other.

Constant practice, both oral and written, should be given in solving the simple types:

$$x-4=12; \quad 2y=18; \quad \frac{z}{2}=6; \quad \frac{3t}{2}=5; \quad \frac{4}{p}=\frac{1}{2}.$$

The pupil must learn to give an explanation of each step in the solution. In order that he may acquire this habit, such explanations must be frequently asked for. He should also be accustomed to check each solution by substitution, even in very simple cases.

In written work it is important that the concluding steps of the solution should not be slurred at first.

E.g. $\quad 15x=45.$

Divide both sides by 15, $\quad x=\frac{45}{15}$

$\quad =3.$

Again, $\quad 144+x=171.$

Subtract 144 from both sides, $\therefore x=171-144$

$\quad =27.$

This serves as a useful check to such an error as: $4x=6$, $\therefore x=2$.

It is important that pupils should be able to give the justification for their steps, but there is a time for the routine mechanical practice of operations without which no facility is gained and work is always uphill. It is easy for an over-keen teacher to make the mistake of constant interruption with the question 'Why?' forgetting that he is disturbing the formation of an essential habit.

There are certain phrases which should be avoided with beginners; at a later stage they may be either harmless abbreviations or harmful, according to the way they are introduced and used.

Two such are: 'bring a term over to the other side (of an equation) and change its sign', and 'cross multiply'.

The conditions that make these harmless are:

(i) that originally they are arrived at as generalisations of the pupil's own experience;

(ii) that while using these for convenience, the pupil is able to explain them if asked to do so.

Thus, he should see that 'bring x over and change its sign' covers the case when x is added to both sides of $a=b-x$, and when it is subtracted from each side of $a=b+x$, and that deducing $ad=bc$ at once from $a/b=c/d$ is only short for multiplying each side by bd. If either of the phrases mentioned is learnt as a 'rule', without being understood, its use becomes seriously harmful. The use of the rule may also lengthen the work, as in solving $x/2=3/5$.

Addition and subtraction of simple algebraic fractions with numerical denominators should be done before solutions of ordinary fractional equations. This should check the tendency to ignore all denominators, a tendency which is greatly fostered by presenting such fractions for the first time in equations.

In fractional equations it is well with beginners to insist on a written explanation of the step by which denominators are removed.

For example: $$\frac{x-2}{5}=\frac{x+1}{7}.$$

Multiply each side of the equation by 5×7. (It is necessary carefully to point out that both sides must be multiplied by this number even when the right-hand is only a number.)

Never allow $\dfrac{7(x-2)=5(x+1)}{35}$, because an equation is a statement and a statement cannot be divided by 35.

Young pupils should not be encouraged to clear of fractions and also to remove all brackets in one step; in the majority of cases this results in mistakes in sign and mistakes in multiplication.

§ 4·2. CHECKS

When testing roots the pupil should realise that he must test the equation as given: to test an intermediate step would not show up a mistake made before the line in which the check is applied.

The correct way of checking solutions of equations is as follows :

The solution of $\frac{x-1}{3}+4=9-\frac{2}{5}(3x-2)$ is obtained as $x=4$.

Check. L.H.S. (left-hand side) $=\frac{4-1}{3}+4$

$$=1+4$$
$$=5.$$

R.H.S. (right-hand side) $=9-\frac{2(12-2)}{5}$

$$=9-4$$
$$=5.$$

Most boys, even very intelligent ones, seem to have an irresistible temptation to use the following arrangement :

$$\frac{4-1}{3}+4=9-\frac{2(12-2)}{5};$$
$$\therefore\ 1+4=9-4;$$
$$\therefore\ \quad 5=5,$$

probably because it suffices to convince them that their solution is correct. They should be brought to realise that as a form of statement it is illogical. The full viciousness of the arrangement becomes apparent when it is applied to the proof of identities.

The following is worse still :

$$5(4-1)+60=135-6(12-2),$$
$$20-5+60=135-72+12,$$
$$75=75.$$

To the fundamental fault already indicated it adds those mentioned in §§ (i) and (iv) below. There is also this further objection, that it would not detect an error made in transforming the original equation to something other than

$$5(x-1)+60=135-6(3x-2).$$

It is worth while occasionally to verify that the equation is not satisfied by a number other than the solution.

The objections to the wrong methods are :

(i) The sign $=$ is used as an expression of hope, not as a statement of fact. In the first line it makes an assumption that something is true, whereas it is to be shown that it is true, and that will only be done when the last step of the check is completed.

(ii) To allow a pupil to state that he has proved that $5=5$ (or $75=75$) is detrimental to his acquiring a habit of logical thinking or logical expression.

(iii) If the method of checking follows the procedure of the solution, the pupil will probably repeat any mistake he has made (such as errors in sign or in clearing fractions). A procedure which is different from that of the solution provides a better check *qua* check; it also tends to confirm the validity of any algebraical processes employed in the solution.

(iv) As soon as the value of x is known, the expressions involved in the equation become arithmetical numbers and cease to require algebraical rules (except the rules of signs). The solution is algebra, the check is arithmetic. Thus the fundamental meaning of a bracket * is emphasised, viz. that it contains symbols which represent a single number. The wrong method misses this.

§ 4·3. SIMULTANEOUS EQUATIONS

These should be introduced by a problem. E.g. Smith, who has a bag of half-crowns and no other coins, owes 1s. to Brown who has nothing but florins. How many coins does he pay to B. and how many coins does B. give him as change?

Let S. pay x half-crowns and receive y florins, that is, the value of what he pays is $5x$ sixpences and of what he receives is $4y$ sixpences;

$$\therefore\ 5x-4y=2. \qquad \text{(1)}$$

A table of values of x and y satisfying this equation can be made:

x	2	6	10	14 etc.
y	2	7	12	17 etc.

and it is seen that the number of solutions is limited only by the number of coins possessed by S. and B. This illustrates the fact that a single equation in two unknowns is indeterminate. Some other condition is necessary to make the solution determinate.

Suppose that the number of coins used in the transaction is 13.

Then $$x+y=13. \qquad \text{(2)}$$

Another table can be made for solutions of (2):

x	13	12	11	10	9	8	7	6 etc.
y	0	1	2	3	4	5	6	7 etc.

* In a fraction, the dividing line is a bracket; the numerator being one number, the denominator another.

The only common solution is $x=6,\ y=7$.

The equations are *simultaneously* (at the same time) true, if and only if $x=6,\ y=7$. I.e. S. pays 6 half-crowns and receives in change 7 florins.

The problem thus approached lends itself to graphical illustration showing the simultaneity.*

Methods of Solution.

Consider the simultaneous pair of statements:

$$5x-4y=6, \quad \text{.................................}(1)$$
$$8x+3y=19. \quad \text{.................................}(2)$$

There are two general methods of solution.

(i) *Substitution.* One unknown is obtained in terms of the other from one equation. This value substituted in the other equation reduces it to a simple equation; thus from (1) $x=\dfrac{6+4y}{5}$.

Substitution of this in (2) gives $\dfrac{8(6+4y)}{5}+3y=19.$

The method is rarely convenient in solving when both equations are of the 1st degree, but it is the standard method when one equation is of the 1st and the other of higher degree, and even with 1st degree equations the method provides a good algebraical exercise.

(ii) *Elimination* (kicking-out-of-doors; Lat. *limen*, the threshold). By suitable multiplication (or division) the equations are put into such a form that the coefficients of one unknown are the same in both equations and the results are added (or subtracted).

At first the pupil should describe his method thus:

To eliminate y.

Multiply both sides of (1) by 3 and get $15x-12y=18$, ...(1*a*)

" " " (2) " 4 " $32x+12y=76$. ....(2*a*)

Addition gives $47x=94$;

$\therefore\quad x=2.$

Substituting in (2) we get $16+3y=19$;

$\therefore\quad y=1.$

Solution $x=2,\ y=1.$

Check in (1) $5x-4y=10-4=6$

(or substitute in (1) and check in (2)).

Later the written description may be dropped.

The following should never be allowed:

$$5x-4y=6\times 3,$$
$$8x+3y=7\times 4.$$

* This does not mean that the problem should be *solved* graphically.

It is worth noting that some boys seem to be taught in an example like the above to multiply each side of (2) by -4 and then subtract, i.e. to change every sign twice over to get them back to what they were originally. This method is bad.

It is occasionally more convenient to obtain the second unknown by elimination instead of substitution. This is the case when the first unknown is a heavy fraction, and especially in literal equations.

§ 4·4. EQUATIONS WITH THREE UNKNOWNS

A few examples of equations in three or more unknowns will help the pupil to realise

(i) that for a determinate solution there must be as many equations as unknowns ;

(ii) why the elimination method is convenient and how it works ;

(iii) the advantage of labelling his equations systematically and arranging his work in orderly fashion.

§ 4·5. QUADRATIC EQUATIONS

For some suggestions about these see § 5·1.

§ 5. FACTORS

§ 5·1. FACTORISATION IN GENERAL, AND THAT OF QUADRATIC EXPRESSIONS IN PARTICULAR

[Diagrams can often be used in the teaching of this section.]

In arithmetic, factorisation appears merely in an incidental fashion —factors are useful in dealing with fractions and the fraction-like expressions which appear in a 'proportion' sum—but there may be no systematic study of the art of factorising numbers.

In school algebra, however, factorisation is apt to bulk very large; it has a chapter or chapters to itself in the books and much time is often devoted to it.

What is the reason for this difference and how far is it justifiable?

Before attempting to answer these questions it may be well to note a fact of experience: there are few if any of our pupils who cannot attain a reasonable degree of success in factorising ordinary trinomials such as x^2-x-6, $2x^2+5x+2$; there are very many to whom practically all other forms seem to present insuperable obstacles. They will manage $7x+7y$ and even $ax+bx$, but the step to $x(2x+1)+7(2x+1)$ seems terribly high. Nor is it easy to get them to realise that the expression last written, especially when it appears as a stage in their own work, is not already factorised.

This experience—and it will hardly be challenged—suggests that the essential difficulty in this part of algebra is that of gaining and developing the sense of *form*. Just as a figure which to anyone with some geometrical training is in itself significant and suggestive may be a mere chaos of lines to a person without such training, so an algebraic expression however expressive to a trained algebraist may be a mere confusion to another.

It is further to be remembered that when a person has once fully gained a power of this sort—the power to walk, to throw a stone, to read a language—he is very apt to forget that there was ever any difficulty and to be surprised that others cannot do such simple things.

If we could get all our pupils to read an algebraic expression as easily as we do ourselves it would be very nice—but experience suggests that with the limited time and energy at their disposal we must not expect too much: better get them to do a little honestly and well than have them floundering in a quagmire of mistakes and confusion.

A further point. In school algebra, at all events as at present conceived, those forms which pupils find easy to factorise ($2x^2+7x+3$, etc.) have an importance peculiar to themselves because of the connection between factorisation and the solution of equations.

The first practical conclusion seems to be that the factorisation of trinomials can and should be mastered by all—and this is not merely possible but also important and useful because of the connection with equations.

Work beyond this is of course essential for anyone who aspires to any real mastery of algebra ; but it is necessary, not because it has applications, but because it is part of the training in the sense of form and may therefore be classed with such things as literal equations, symmetric functions and so on. A considerable number of boys find great difficulty in ' seeing ' that $x(2x+1)+7(2x+1)$ and $(a+2b)(2x+1)+(2a+b)(2x+1)$ are complex binomials of the same form as $pa+pb$. This does not necessarily mean that they have no sense of form, but only that their eyes open slowly. Most of these boys after a first essay that has seemed barren of result will return to the subject later (e.g. after solving quadratic equations) and find that they have developed a clearer vision. Then, and only then, will they be able to factorise intelligently such expressions as $p^2mn+pqm^2+pqn^2+q^2nm$.

Trinomials. To factorise $2x^2+13x+15$ we might proceed by splitting up the $13x$:

$$\begin{aligned} 2x^2+13x+15 &= 2x^2+3x+10x+15 \\ &= x(2x+3)+5(2x+3) \\ &= (2x+3)(x+5). \end{aligned}$$

This may be scientific ; but it introduces a difficulty if there is truth in the preceding exposition.

We therefore definitely recommend adherence to the method which is probably generally practised : the $2x^2$ suggests $2x$ and x as initial terms of the factors ; for the second terms possibilities are 5 and 3, 3 and 5, 15 and 1, 1 and 15 ; that soon shows that $(2x+3)(x+5)$ gives the correct middle term.

However, a considerable minority of teachers advocate the method of splitting up the middle term, and it may be true that a considerable minority of pupils respond to it more readily.

All teachers should therefore remember that, whichever method is adopted, it will probably be desirable to try the other with those boys who are not being successful.

It is essential to secure that the factors are right, even if this involves (for the slower people) doing the multiplication out at length : to fail to obtain factors may be venial, but to give wrong factors is criminal.

Stress should be laid on the general truth that if one factor is known the other factor can be obtained by division; this provides a useful variant for checking and the incidental practice in division is very valuable.

In most cases perhaps when introducing a new subject or rule it is advisable to begin with the simplest possible cases; but there are exceptions, and there is such an exception here. It is definitely inadvisable to give many examples of the type x^2+7x+6 before introducing such as $2x^2+7x+3$. A better conception of the suitable type for beginning is perhaps that in which all the signs are positive.

Among the examples should be included some—not a few, but many—which will not factorise. The fact that some expressions like some numbers will factorise while others will not, is itself important; but also the mixture makes it easier to prevent mere guessing without verification.

The prejudice in favour of the x^2 term coming first should not be allowed to grow up; forms like $2+5x-12x^2$ should be freely used, and without alteration of the order, or else there will arise a fruitful source of mistakes.

With trinomials in a single letter there will of course be taken homogeneous expressions in two letters, and stress should be laid on the necessity for homogeneity in the factors.

Solution of Equations. The main use of factors in arithmetic is for the addition and simplification of fractions, and of course work of this sort is customary in algebra. It has great value as practice in manipulation, and it ought to be so treated as to give a fuller grip of the principles involved in dealing with fractions. But the really important use of factors, as stated above, is for the solution of equations other than those of the first degree.

Here we have a *general* principle of the highest importance, and it should be presented as such, not merely as a special dodge for solving quadratics.

In its general form the principle is just as simple and easy to grasp as in the special form where there are only two factors—perhaps easier because more impressive: viz. if the product of a number of expressions is zero, one of them must be zero; or in symbols: if $P\times Q\times R\ldots=0$, then either $P=0$ or $Q=0$ or $R=0\ldots$.

This should be contrasted with the apparently parallel case of addition: if $P+Q+R+\ldots=0$, no conclusion can be drawn as to P or $Q\ldots$.

In solving, e.g. $3x^2-10x-8=0$, by factors, the step in which this principle is used should on no account be omitted by beginners.

$$(3x+2)(x-4)=0;$$

$$\therefore\ \textit{either}\ 3x+2=0\ \textit{or}\ x-4=0.$$

Other examples of the principle should be included: e.g.

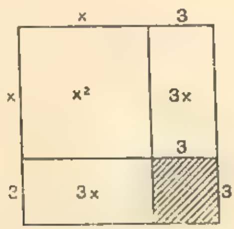
If $xy=0$ and $x=1$, what is y?
If $(a-b)x=0$, can you say anything about x?
If $xy=3$, can you say anything about x?
If $(x+1)(y-2)=0$ and $x=-1$, can you say anything about y?

And it is valuable—and quite easy—to take some cubics which can be solved by factorisation.

Completing the square is the method to be used normally for solving quadratic equations where factors cannot easily be found.

It is probably best to reduce the coefficient of x^2 to unity at the outset.

Practice is necessary in the art of finding what to add to make, e.g., x^2+6x up to a perfect square. Some like a geometrical method as indicated in the figure:



$\therefore$ add 9.

But it comes to the same thing, and maybe with less trouble, to recognise that x^2+6x is the first two terms of the square of $x+3$ which when complete is x^2+6x+9; $\therefore$ add 9. The difficulty becomes serious of course only when the coefficient of x is odd, or, worse still, is a fraction.

In any case the rule 'add the square of half the coefficient of x' if used at all should only be used as the outcome of much practice in one or other of the methods given above—or any other as good or better which a teacher may find for himself.

In solving quadratics there are two possible procedures after reaching the stage

$$(x-a)^2=b.$$

(i) $(x-a)^2-b=0$;
$\therefore\ (x-a+\sqrt{b})(x-a-\sqrt{b})=0$;
$\therefore$ either $x-a+\sqrt{b}=0$
or $x-a-\sqrt{b}=0$,
etc.

(ii) $(x-a)^2=b$;
$\therefore\ x-a=+\sqrt{b}$ or $-\sqrt{b}$;
$\therefore\ x=a+\sqrt{b}$ or $a-\sqrt{b}$.

Of these (ii) is probably that generally used and rightly so. As usual in such matters, no one who has habituated himself to (i) should be pressed to change; and (i) shows more clearly the converse process of using the solution of the equation to determine the

factors of the expression. The two methods may therefore be demonstrated, compared and shown to be equivalent.

Solution by the formula is quite unimportant in elementary work. It is not recommended for beginners, and it has been found to be quite unsuccessful as a cramming device. It should be allowed only if the user when challenged can do without it.

Graphical Solution. Having solved easy equations as above, boys may be shown how quadratics can be solved graphically; an equation like $2x^2-5x+2=0$ with factors should first be used; then others, such as

$$2x^2-5x=11, \quad 2x^2-5x-2=2x+5,$$

can be solved by the same graph, though they may not be solvable by factors.

It is important to notice that there may be no solutions; here boys should be taught to say that there are none, and not to pretend that there are two solutions called imaginary.

Questions on inequalities should not be neglected even though the available time is short. They arise naturally out of graphs as do their answers, but they should then be examined also from the algebraic point of view: e.g. a graph makes it clear that x^2-5x+6 is negative for values of x between $x=2$ and $x=3$. This should also be seen by considering the expression in factors.

Similarly, graphs lead naturally to questions about maximum and minimum values; and the resulting algebraic discussion reveals the utility of the process of completing the square in a field wider than that of its original use.

There is much variety in the miscellaneous questions which can be devised under this heading: e.g.

(i) Fill in the blank in $x^2+2x+13=(x+1)^2+\ldots$ and give the smallest value of $x^2+2x+13$.

(ii) Show that x^2+x+1 is never negative.

(iii) What is the maximum value of $8-6x-x^2$? Has it a minimum value?

Such questions provide a real test as to whether a boy is working by rote or with intelligence.

Other Types of Factorisation. The forms $a^2\pm 2ab+b^2$, a^2-b^2 are of course included in the general trinomial form, but should be learnt by heart and should be applied, especially the last, to suitable arithmetical cases. Elaborations of these forms, e.g. $x^2-(y+z)^2$, etc., fall from our present point of view, with others such as $a^2+ab+ac+bc$, $b^2-c^2+(b-c)^2$ under the general class of factors which are studied for the sake of algebraic form, not for their direct utility, and in dealing with which the weakest pupils may have to admit failure.

There is, however, one practical application of such factors, not an easy one, but which, if it can be reached, may be taken as a suitable terminus for the study of factors in the case of pupils of average ability. This is the work which obtains the formula $\Delta=\sqrt{\{s(s-a)(s-b)(s-c)\}}$ for the area of a triangle, and although this is a rather long piece of consecutive work it needs nothing more difficult than the factors of $a^2-b^2+2bc-c^2$.

Most of such factorisations depend in one way or another on grouping terms into units themselves composite, e.g.

$$a^2+ab+ac+bc=a(a+b)+c(a+b),$$
$$x^2-y^2+2y-1=x^2-(y-1)^2, \text{ etc.}$$

This preliminary process in itself often presents difficulty, and there ensues the further difficulty of thinking of and treating the composite terms in precisely the same fashion as if they were simple.

One of the characteristic features of algebra is, of course, just this following of a known rule, familiar in simple applications, in cases which look more and more alarming: $\frac{1}{3}+\frac{1}{7}=\frac{7+3}{21}$ is familiar and easy; $\frac{1}{a}+\frac{1}{b}$ when new is a little harder; $\frac{1}{x+2}+\frac{1}{x+3}$ is harder still, and so on. As the expressions get more elaborate, the sense of meaning disappears, and mistakes will multiply unless the basic rule is clearly and strongly grasped.

Some attempt should naturally be made to get all pupils over the earlier stages of these extensions, but great care is required if the work is not to do more harm than good: each pupil's work must be often checked in detail, for it easily happens, especially, for instance, in factorising such an expression as x^2-y^2+2y-1, that two bad mistakes neutralise one another and a correct result is obtained though the work is worthless. The amount attempted should therefore be carefully graded to the capacity of the class, and there may well be cases where this part of algebra should be abandoned altogether.

A few practical suggestions may be made:

There is a very useful rule not so generally known as it should be: if the expression to be factorised contains a letter *which occurs to the first power only*, group all terms containing that letter into one term, all the rest into another; then the common factor of the two terms, if there is one, is a factor of the whole.

E.g. to factorise $a^2-ab-ac+bc$.

b and c occur to the first power only; either of them will serve our purpose; choose either, say b,

then
$$\begin{aligned} E &\equiv (a^2-ac)-(ab-bc) \\ &= a(a-c)-b(a-c) \\ &= (a-c)(a-b); \end{aligned}$$

on the other hand, if $E \equiv a^2 - ab - ac - bc$

$$= a(a-c) - b(a+c),$$

the two terms have no common factor and E will not factorise.

The following is an example of a very common mistake in factorising by the grouping method :

$$a^2 - ac - b^2 + bc = a(a-c) - b(b-c)$$
$$= (a-b)(a-c)(b-c).$$

The best way of preventing this is to insist that at the final stage no factor must be written down until the groups display a common factor, and when they do, the common factor should be written down first :

$$a^2 - b^2 - ac + bc = (a+b)(a-b) - c(a-b)$$
$$= (a-b)(\qquad).$$

In an example of this kind the degree of the wrong answer betrays the error at once to the pupil taught to consider the degree of algebraical expressions.

Great stress should be laid on the use, and the accurate use, of the word 'term'. The expression $x^2 - y^2 + 2y - 1$ has four terms, but by grouping as $x^2 - (y-1)^2$ we reduce it to two terms and have a form to be treated precisely like $a^2 - b^2$.

When an expression is factorised it has been reduced to a single term. The final process indicated by the form is a multiplication, whereas in the intermediate steps leading to factorisation the process indicated is addition or subtraction.

§ 5·2. FACTORS OF FOUR-TERM EXPRESSIONS

In dealing with factors of the preliminary types $px + py$ and $a(c+d) + b(c+d)$ the first thing is to make the boy realise that he is dealing with a binomial expression and that there is a factor common to the two terms (a teacher has suggested calling this the 'shouting' factor); the next thing is to make the boy write down this common factor immediately and then see how many times it goes into each term.

Thus $px + py$ has a common factor p, so the boy writes

$$px + py \equiv p(\qquad) \text{ and } px + py \equiv p(x+y).$$

Again, when he meets $a(c+d) + b(c+d)$ he writes at once

$$a(c+d) + b(c+d) \equiv (c+d)(\qquad)$$

and $$a(c+d) + b(c+d) \equiv (c+d)(a+b).$$

Before going on to written examples it is useful to take a whole exercise *viva voce*, merely naming the 'shouting' factor in each.

The following sequence has also been found successful :

(i) In dealing with factors of the preliminary type $ax+ay$ include some expressions involving capital letters :

$$\text{e.g.}\quad P+\frac{PRT}{100},\ aW-bW,\ Mg+mg.$$

(ii) In the first lesson on the next type $a(c+d)+b(c+d)$, argue as follows :

'We know that $ax+bx$ can be factorised, $ax+bx\equiv x(a+b)$.

'Replace the simple common factor x by something more complicated, say the binomial $(c+d)$.

*'We should expect $a(c+d)+b(c+d)$ to be equal to $(c+d)(a+b)$.

'To see this we write

$$a(c+d)+b(c+d)$$

$\equiv aB+bB$ replacing the *bracket* by a single letter (hence the large B), which is the 'shouting' factor mentioned above,

$\equiv B(a+b)$

$\equiv(c+d)(a+b)$.'

Then one can go on to other illustrations, complications of sign, absence of brackets, or groupings. But, in the first examples worked on the board, writing a large letter for the common factor is a sound device for emphasising the essential step, from $a(c+d)+b(c+d)$ to $(c+d)(a+b)$. Additional emphasis can be given by writing the common factor in coloured chalk.

In written examples one may write, e.g.

Factorise $pq-ps+qr-rs$.

$$\begin{aligned}\text{Expression}&\equiv p(q-s)+r(q-s)\\&\equiv pB+rB \text{ if } B\equiv(q-s)\\&\equiv B(p+r)\\&\equiv(q-s)(p+r),\end{aligned}$$

though this setting-out can be abbreviated to the usual form as soon as the processes have become mechanical.

§ 5·3. THE FACTOR AND REMAINDER THEOREMS

In solving quadratics, e.g. $3x^2-4x-4=0$, i.e. $(x-2)(3x+2)=0$, the pupil has learnt that since $x-2$ is a factor of $3x^2-4x-4$, $\therefore$ the substitution of 2 for x makes $3x^2-4x-4$ equal to 0.

The converse of this argument is that if the substitution of 2 for x makes $3x^2-4x-4$ equal to 0, then $x-2$ is a factor of $3x^2-4x-4$.

This is a case of the Factor Theorem (often erroneously called the

* Compare Appendix A 5.

Remainder Theorem), which states that, if $f(x)$ is a polynomial, and if $f(a)=0$, then $x-a$ is a factor of $f(x)$.

The pupil's experience in solving the quadratic by factors has prepared an approach for the Factor and Remainder Theorems, and will help to put the new work on to familiar ground. Any formal proof of the Remainder Theorem should be left till quite late in the algebra course, but it is easy to give an informal explanation of the Factor Theorem and to get boys to apply it to simple examples at a much earlier stage. For example, it is easily seen that

$$13x^2-27x+14$$

is equal to 0 when $x=1$; hence $x-1$ is a factor of the expression. Easy questions like

(i) Guess a factor of x^3+4x^2+x-6;

(ii) For what value of c is $x^2+11x+c$ divisible by $x-2$?

(iii) For what value of n is $x+4$ a factor of x^3+x^2+nx+8?

(iv) For what values of b and c is $2x^3-3x^2+bx+c$ divisible both by $x-2$ and $x-4$?

are appropriate.

There is an intermediate stage in which a proof of a special case may be given. Thus, from the division sum

$$\begin{array}{r|l} & x^2+5x+14 \\ \hline x-2 & x^3+3x^2+4x+5 \\ & \underline{x^3-2x^2} \\ & 5x^2+4x \\ & \underline{5x^2-10x} \\ & 14x+5 \\ & \underline{14x-28} \\ & 33 \end{array}$$

follows:

$$(x^2+5x+14)(x-2)+33 \equiv x^3+3x^2+4x+5,$$

an identity which the boy should verify by sheer multiplication; he can then be made to see that it is enough to write

$$\begin{array}{r|l} & x^2+\text{ etc.} \\ \hline x-2 & x^3+3x^2+4x+5 \\ & R \end{array}$$

i.e. $$(x^2+\text{ etc.})(x-2)+R \equiv x^3+3x^2+4x+5,$$

and then to put 2 for x, getting

$$R=2^3+3.2^2+4.2+5.$$

The actual division of bx^3+mx^2+nx+p by $x-a$ is useful. The boy who is ripe for 'literal' work of this kind should also be ready for a more general proof.

It is necessary to emphasise that R must be a constant (i.e. independent of x). Example (v) in the following list of applications is useful in driving the point home.

General Proof. There is a choice of methods :

(A) If $q(x)$ is the quotient and r the remainder when the polynomial $f(x)$ is divided by $(x-a)$,

$$(x-a)\,q(x)+r\equiv f(x).$$

That this is true, and the fact that it is an identity, will best be realised by a boy who has done the verification in a numerical example as suggested above. Putting $x=a$ gives $r=f(a)$.

(B) The identity $x^m-a^m=(x-a)(x^{m-1}+x^{m-2}a+\dots a^{m-1})$ is easily verified, and it shows that x^m-a^m is divisible by $x-a$. Any polynomial

$$\begin{aligned} &x^n+Ax^{n-1}+Bx^{n-2}+\dots+Kx+L \\ &=(x^n-a^n)+A(x^{n-1}-a^{n-1})+\dots+K(x-a) \\ &\qquad +a^n+Aa^{n-1}+Ba^{n-2}+\dots+Ka+L \\ &=\text{a multiple of } (x-a), \text{ plus} \\ &\qquad a^n+Aa^{n-1}+Ba^{n-2}+\dots Ka+L\,; \end{aligned}$$

this proves the theorem.

The mathematician will probably prefer method (A), but there are some dangers connected with it ; if it is presented to an immature class, it will probably be learnt without being understood.

The remainder theorem and the functional notation ought not to be introduced at the same time, because the average boy does *not* appreciate the meaning of $f(x)$ instantaneously. Moreover, it is advisable to avoid giving the false impression that $f(x)$ always denotes a polynomial.

Amongst the applications of the Remainder Theorem which belong to the later stage may be mentioned examples like :

(i) Show that $b-c$ is a factor of $\Sigma a^2(b-c)$.

(ii) Factorise $\Sigma a^3(b-c)$.

(iii) Show that a, b and c are factors of

$$\Sigma a^4-\Sigma(b+c)^4+(\Sigma a)^4.$$

(iv) Show that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$.

(v) Find the remainder when a polynomial $f(x)$ is divided by $(x-a)(x-b)$.

(vi) Why is it permissible in method (A) to put $x=a$ after dividing by $x-a$ although division by zero is impossible ?

§ 6. FRACTIONS

Text-books on elementary algebra published in the last century often contained a chapter entitled 'Harder Fractions'. These contained examples of a complicated nature involving heavy and comparatively useless manipulation. Drill in such examples was regarded as necessary in order to 'train the mind' and defeat the examiner. Recently both examiners and writers of text-books have taken a more reasonable line, but it is still desirable to warn teachers and examiners against the dangers of indulging in an orgy of manipulative work for its own sake.

Fractions with numerical and single-term denominators should be introduced naturally, and, in fact, they cannot be avoided in the early work with formulae. The rules for dealing with them in algebra are essentially the same as the arithmetical rules, and many of the same teaching principles have to be observed. (See *Report on the Teaching of Arithmetic in Schools*, § 9.) The only additional point is dealt with in (ii) below. It need not even be suggested to the boy that there is anything fresh to learn about them when he first meets them in algebra.

Two special cautions will certainly have to be given when the need for them arises:

(i) In adding fractions the common denominator must not be omitted. There is an almost ineradicable tendency in boys to omit it once they have become accustomed to the practice of multiplying both sides of a fractional equation by a number so that the denominators disappear. This omission in adding fractions should not be regarded as a venial error.

(ii) The bar of a fraction has the effect of a bracket. Consequently $3-\frac{x-1}{2}$ becomes $\frac{6-(x-1)}{2}$ and eventually $\frac{6-x+1}{2}$. It is probably better to write down both these stages in simplification until the boy is thoroughly familiar with the idea.

Once these two points have been clearly understood, boys will manipulate fractions with as much accuracy as can be expected from human frailty.

Fractions whose denominators contain more than a single term will occur in problems (e.g. in which trains increase their speeds), if nowhere else, and it is often found necessary to do some purely

manipulative work on such fractions. This should be done after factors have been tackled, and provides a better method of increasing the boy's grip on factors than a constant reiteration of the order 'to factorise'.

The method of dealing with fractions in algebra differs very little from that used in arithmetic, but one new difficulty arises and some cautions should be given.

The difficulty is the relation between expressions of the form $A-B$ and $B-A$. This occurs, for example, in simplifying

$$\frac{x}{x-y}-\frac{y}{y-x}.$$

It may be noticed that a negative algebraic fraction may be written in various ways, e.g.

$$\frac{-a}{b}\equiv\frac{a}{-b}\equiv-\frac{a}{b},$$

and of these three forms the last is to be preferred, for it takes the fraction as a whole and states that it is negative.

Thus it is important to realise that

$$(x-2)(x-3)\equiv(2-x)(3-x)\equiv-(x-2)(3-x)\equiv-(2-x)(x-3),$$

and knowledge of this point may be tested by such questions as:

'If $x-y=7$, can you say anything about the values of $2x-2y$, $y+x$, $y-x$, $y-2x$ ... ?' and 'simplify where possible:

$$\frac{a+b}{b+a},\ \frac{a-b}{b-a},\ \frac{a-b}{b+a}.\text{'}$$

In dealing with fractions in which this difficulty arises, denominators should be arranged in either ascending or descending powers of some letter.

Thus
$$\frac{1}{x^2-3x-2}-\frac{1}{2x-x^2}+\frac{1}{3-x^2-2x}$$
$$\equiv\frac{1}{x^2-3x-2}+\frac{1}{x^2-2x}-\frac{1}{x^2+2x-3}.$$

The following points, although not new, need emphasis:

(i) In adding fractions in arithmetic the extra work entailed through using *some* common denominator other than the lowest is not often considerable. In adding algebraic fractions failure to factorise *all* denominators and to work with the *lowest* common denominator may entail great labour.*

(ii) In a simplification it sometimes saves labour to group the terms rather than find the common denominator,

$$\text{e.g. }\left\{\frac{1}{x-1}-\frac{1}{x+1}\right\}+\frac{2x}{x^2-1}\equiv\frac{2}{x^2-1}+\frac{2x}{x^2-1}.$$

* For an example of this see § 9, p. 54.

Again, with equations involving fractions it sometimes saves time if each side is simplified first,

$$\text{e.g. } \frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3};$$

$$\therefore \frac{1}{(x-2)(x-1)}=\frac{1}{(x-4)(x-3)}.$$

(iii) As in arithmetic, when fractions have been expressed with the common denominator, the numerator should be inspected for factors in case the expression will simplify. Those who set manipulative examples often arrange the questions so that reduction to lower terms is possible.

(iv) The converse process of splitting up a fraction into partial fractions is at least as important in later applications and should be tackled before a heavy course of the gathering process is undertaken. The importance of this process lies in post-certificate work.

(v) Arrangement in cyclic order facilitates some examples, but this is specialist work.

A final caution. A certain amount of drill in the manipulation of fractions is necessary, but care should be taken that it is not overdone. It should be remembered that real proficiency here may indicate a future mathematical specialist, and that mastery by all members of the class can be obtained only by an expenditure of time which would be *much* better spent on other parts of the subject.

§ 7. GRAPHS

[Teachers are reminded that the school course need not follow the order of the following paragraphs.]

(·1) Whether graphs are regarded in the first instance as part of the arithmetic or the algebra course is immaterial, but it is essential that considerable attention should be devoted to the reading as well as to the drawing of graphs from purely arithmetical data before any algebraic notation is introduced.

This arithmetical work (see *Report on the Teaching of Arithmetic*, p. 54) should eventually lead up to the construction of graphs in which the table of values is obtained by calculation or from some simple experiment, like measuring the length of a rubber cord, from which various weights have been suspended. The law implied in this experiment, or in a graph exhibiting the square roots of various numbers, can now be explicitly stated as an algebraic law of the type $L=a+bW$ or $y=k\sqrt{x}$. Some time should be spent upon concrete examples leading to graphs of the type $\frac{k}{x}$ (e.g. time taken to travel a given distance at varying speeds). This eventually leads to a sound understanding of asymptotes, whether the term itself is introduced or not.

The ideas of 'independent variable' and 'dependent variable' follow naturally from such experimental work, though the actual terms may well be postponed until a later stage. The convention that the values of the independent variable are always plotted from left to right across the page must be emphasised from the first, even if the phrase is not used. The pupil can be taught to see that one set of numbers is chosen, and the other set found by experiment or calculation from the first set.

A varied notation should be used, and the variables should not always be denoted by x and y, though for brevity x is usually employed to denote the independent variable in this section of the Report.

As the average pupil takes a considerable time to draw a graph, it is desirable that practice in reading graphs should be given by providing graphs ready drawn upon a blackboard, if none are available in the text-book in use; a number of questions can then be asked about the graph to train the boy's power of reading it.

It is more important that a boy should begin to think of such graphs as pictures of the variation in value of a function than that he should associate them with equations. Anyhow the phrases 'the graph of an equation' or 'the graph whose equation is' are unsuitable at this stage. They belong to a different set of ideas and should be deferred. A boy should think of the graphs of the functions $a+bW$ and $\sqrt{x}$ rather than of the graphs of the equations or formulae $L=a+bW$ and $y=\sqrt{x}$. He should regard the L and the y as convenient but unnecessary abbreviations.*

Labelling of axes. In the early work the nature of the variables will be indicated by writing the quantities and units, e.g. 'Weight in lb.', 'Time in seconds', parallel to the axes. Graphs of functions should have their axes labelled by writing, say, x or W on the axis across the paper and the expression for the function, say, $x(x+2)$ or $a+bW$ along the perpendicular axis. The full expression for the function should be written down until the functional aspect of graphical work is thoroughly familiar. It will then be convenient to use single letters as abbreviations.

Scales. In the arithmetic course a boy should have learnt the necessity of a scale, and the convenience of using as large a scale as his paper permits. He should have been trained to use and to select scales which enable intermediate readings to be rapidly determined, and to avoid using a scale division to represent 3 or 7 units of a variable. To discover his scales he will have been instructed to compare the range of the quantities to be represented with the dimensions of his paper.

In the first graphs of functions the range of values of the independent variable will be chosen by the teacher, and the consequent range of values of the function will be calculated by the class. Only when these ranges are *both* determined is it possible to choose scales.

To choose scales to the best advantage requires a good deal of experience and practice. This discipline should not be withheld from the boy by the teacher doing the work for him. At first help—or better, discussion—is needed, but as soon as possible boys should be left to choose their own scales. It is true that in public examinations the scales are often prescribed. This should be regarded as an arrangement to facilitate marking, not as an indication of the proper teaching procedure.

Eventually it is necessary for a boy to produce, unaided, a satisfactory graph in answer to such a question as 'Draw the graph of $(x^2-1)/x$—no range or scale being suggested. Some attention should therefore be given to making freehand sketches of graphs of functions. Boys should be encouraged to make a preliminary free-

* It is possible to carry graphical work up to this stage without the use of negative or of directed number.

hand sketch before attempting the finished graph. Such freehand sketch graphs drawn rapidly on ordinary paper have many advantages in later work.

(·2) *Graphic Solution of Linear Simultaneous Equations.* A customary exercise at an early stage has been the construction of graphs to solve linear simultaneous equations. Graphs are useful to illustrate the uniqueness of the solution, but their use for other purposes raises several points.

(*a*) The main purpose of graphs is not the solution of equations but the exhibition of the behaviour of functions. The first necessity for a boy is to realise that with sufficient patience he can make a graph of any single-valued algebraic expression and that every such expression leads to an orderly curve. Before any graphic solution of linear equations is shown, the general idea of graphing functions of several types should be developed. (See § (·3) below.)

(*b*) Since a boy will have already learnt the more speedy, and certainly more reliable, algebraic method of solution it is a waste of time to teach graphic methods, as they can give only approximate solutions. If the coefficients in the equations are awkward, or numerically large, the graphic method may be preferable to the algebraic, but such equations will not enter into the first course in algebra.

(*c*) Graphic solution is most effectively introduced to solve equations (e.g. cubics) which cannot be tackled by any algebraic process known to the pupil.

(*d*) The solution of a pair of simultaneous equations by finding the intersections of the graphs of two implicit functions is a difficult idea to grasp. It is, in fact, the last stage in the graphic solution of equations. (See below, § (·4).)

For these reasons the Committee feels that the graphic solution of linear simultaneous equations should be excluded from the early stages.

(·3) *Graphing of Functions.* Experienced teachers differ in their choice of graphs to be drawn to develop the idea of graphing of functions. There is certainly no necessity to begin with linear forms, and there are certain positive disadvantages in so doing: they are less interesting than curves; the temptation to guess the general form from insufficient data is stronger; and the correspondence of a straight line to a linear form is much less striking if it appears in isolation than if it appears in contrast to a background of wider experience. Nor is there any need to begin with the comparatively colourless graph of x^2. Such forms as $x(x-2)$ or $x(x+2)(x-4)$ or $120/x$ or $(x-3)/(x-2)$ may with advantage be taken at once.

The early exercises should be chosen so as to exhibit the greatest variety of orderly curves, subject to the condition that the arithmetic involved in calculation must not be the major part of the work. The first quadratic or cubic functions can be given in their factorised forms, and it should be pointed out that these functions are then in the most convenient form for computing their values. The arithmetic in the graphing of, say, $(x-3)/(x-2)$ does not present any serious difficulty, and gives excellent practice in representing fractional values on paper.

There should be no attempt to emphasise the graph of the quadratic function until some such graphs as those suggested have been drawn.

Later the graph of the quadratic function should be developed gradually, beginning with the graph of x^2, and a number of instances should be given before any general connection between the resulting parabolas is introduced. (See below, § (·6).)

Computation of Values. From the earliest stages attention should be given to developing economy of effort in the computation of values of a function. The teacher will, of course, suggest the range of values for x. At first only integral values of x will be used for calculation, the resulting points will be plotted and the choice of intermediate values will be determined by the further points required to fix the curve. It should be emphasised that the *minimum* number of values in the table is the ideal; and that the graph and not the table is the important aim.

Where a table is required it should be planned intelligently before any calculations are done. The expression for the function should be factorised if possible—e.g. use $x(x-4)$ instead of x^2-4x, since $x(x-4)$ is easily evaluated for a series of values of x by writing down these values and underneath the values of $x-4$, and then calculating a row of values for $x(x-4)$.

Various ways of tabulating the evaluation of a general quadratic function, say $3x^2-5x-7$, are in use. The most natural plan is:

$x = -4, -3$	. . . .
$3x^2 = 48$	. . . .
$-5x = 20$	. . . .
$-7 = -7$	. . . .

$\therefore\ 3x^2-5x-7=61$, etc.

If it is possible to calculate mentally the values of the function, it is desirable that this should be done and the graph drawn without writing down any 'table'.

The following method of mental substitution presents little difficulty, and is applicable later to the use of the Factor and Remainder Theorem:

To substitute a number (say p) for x in $2x^3-5x^2-9x+18$.

Multiply the leading coefficient by p, getting		$2p$
add the next coefficient,	„	$2p-5$
Multiply the result by p,	„	$2p^2-5p$
add the next coefficient,	„	$2p^2-5p-9$

and proceed thus by alternative multiplication and addition, getting $2p^3-5p^2-9p$ and finally $2p^3-5p^2-9p+18$.

Substituting 2 for x, one gets stage by stage 4, -1, -2, -11, -22, and finally -4.

When the graph of a function has been drawn, a large number of questions should be asked about the graph: e.g. Has the graph any type of symmetry; has the function a maximum or minimum value; between what values of x is it positive, greater than 2, less than 2, greater than -2, less than -2; for what range is the function increasing with x, etc. Simple questions upon maxima and minima of functions may be solved graphically before graphs are used for the solution of equations.

At a later stage graphs are useful to illustrate the solution of inequalities, e.g. $x^2+4x > 2$.

(·4) *Graphic Solution of Equations.* There are several stages in this work:

(i) At first the intersections of a graph of a function with the axis of x will be used to solve an otherwise intractable equation.

(ii) When such a graph has been drawn for the solution of an equation many new equations can be solved by considering the intersections of the same graph with lines parallel to the axis of x. E.g. after drawing x^3+3x+1 we solve $x^3+3x+1=0$, $x^3+3x+1=2$, $x^3+3x=2$.

(iii) The next stage is to suggest that we could solve other equations—e.g. $x^3+3x=2x+1$, by cutting the graph by a line inclined to the axis of x.

Any equation may be written in a variety of ways:

$$x^3-4x-1=0 \quad\ldots\ldots(1)$$

may be written

$$x^3-4x=1, \quad\ldots\ldots(2)$$

$$x^3=1+4x, \quad\ldots\ldots(3)$$

$$x^2-4=1/x; \quad\ldots\ldots(4)$$

these would be solved graphically

(*a*) by plotting x^3-4x-1 and finding where the expression $=0$;

(*b*) by plotting x^3-4x (i.e. $x(x-2)(x+2)$) and finding where the expression $=1$;

(*c*) by the intersection of the graphs of x^3 and $1+4x$;

(*d*) by the intersection of the graphs of $(x-2)(x+2)$ and $\frac{1}{x}$.

The same solution should be obtained from each variation, and the pupil will gain in experience and judgment from doing such an exercise. It will help him to see that by putting equations into a suitable form he can use a template of the graph of x^3 to solve any cubic, and one of the graph of x^2 to solve any quadratic equation or even a cubic. Before finishing such work he should realise that any equation in one unknown can be solved graphically.

(iv) The final stage is reached when two functions are defined implicitly, e.g. by $x-y=1$ and $x^2+y^2=4$, and their simultaneous values can be found. Such work should only be incidental: its systematic treatment belongs to analytical geometry.

(·5) *Asymptotes.* At an early stage some time will have been spent upon concrete examples leading to graphs with asymptotes. When graphs of functions with asymptotes parallel to the y-axis are drawn, e.g. $(x+2)/(x-1)$, it should be clearly stated that the value of the function does not exist when $x=1$. (Boys sometimes misinterpret 'does not exist' to mean 'has no value and is therefore zero'.) There is a break on the graph. The statement that 'the function is equal to infinity' should certainly not be used. Nor should the word 'infinity' be used until the boy is ready to understand what is meant by saying that it is a purely conventional method of describing a certain type of discontinuity in the graph. It is rare to find a class of boys all of whom are entirely ignorant of the word 'infinity' and its symbol; usually they are proud of their half knowledge, and it may be better to lighten the darkness of their mind by using some such phrase as 'near $x=1$ the function becomes immeasurably great' or 'as big as you please' than to attempt logical precision of statement.

(·6) At a fairly late stage a boy should summarise his conclusions on the connection between the form of an algebraic relation and the shape of the curve to which it gives rise. If the plotting of functions has not been merely mechanical, he will have compiled (mentally at any rate) a summary of gèneral ideas that are at his disposal, such as:

(i) that the shape of the graph depends on the degree of the function, e.g. for an integral function of the nth degree the graph has $n-1$ 'bends' and may be cut in n points by a straight line;

(ii) that, if $x-a$ is a factor of the function, the graph cuts the x-axis where $x=a$;

(iii) that, if $x-a$ occurs as a factor of a denominator in the function, there is no point on the graph for the value $x=a$, but there is an asymptote.

The idea (i) may be used as a starting-point for the study of graphs forming a family. For instance, a boy should soon be able to see the connection between members of the family $y=kx$, for various values of k, and between members of the family $y=3x+c$ for various values of c. A group of graphs belonging to such a family should be drawn with the same scale and axes, so that the connection becomes obvious before it is explicitly stated.

The family $y=Ax^2+Bx+C$, i.e. the graph of any quadratic function may be approached by first studying the family

$$y=a(x-b)^2+c.$$

The relation of this to $y=x^2$ is not difficult if the work is taken in easy stages, in which we sketch successively x^2, x^2+4, $(x-3)^2$, $(x-3)^2+4$, and then $ax^2\,(a>0)$, ax^2+b, $a(x-b)^2+c$. Then the case $a<0$ may be taken. It is important not to obscure the argument by elaborate calculations of values.

At this stage the connection between the graph of a function (e.g. x^2) and the inverse function (e.g. $\pm\sqrt{x}$) might be shown.

(·7) The idea of the gradient of a straight line and then the idea of the gradient of a curve at a particular point may now be brought out. This gradient can then be found by drawing a tangent, and the calculation of the gradient of a graph is a definite preliminary to the study of the Calculus.

(·8) *The use of an auxiliary graph in plotting.*

For example, to plot $\dfrac{2x^2+1}{x-1}$.

This $$\equiv 2(x+1)+\frac{3}{x-1}.$$

Plot the linear function $2(x+1)$ and add to the ordinates the values of $\dfrac{3}{x-1}$.

Again, to plot $\dfrac{x^3+1}{x}$, i.e. $x^2+\dfrac{1}{x}$, plot the parabola x^2 and add to the ordinates the values of $\dfrac{1}{x}$.

The method simplifies the arithmetic of substitution, enables a sketch graph to be rapidly drawn, and so gives control and facilitates the discussion of asymptotes.

Similarly in trigonometry a good way of plotting $3\sin x+5\cos 2x$ is to plot $3\sin x$ and $5\cos 2x$, using the same axes and units, and add the ordinates *in situ*. For pupils who retain a mental picture

of the graphs of the trigonometrical functions—as all pupils should—this provides a rapid method of drawing sketch graphs suitable for solving equations and investigating maxima and minima.

E.g.
$$\sin x \cos^3 x = \tfrac{1}{4} \sin 2x(1 + \cos 2x)$$
$$= \tfrac{1}{4} \sin 2x + \tfrac{1}{8} \sin 4x$$

lends itself to this treatment.

This compounding of graphs, which is much used in practical work, is of value also as a preparation for understanding the superposition of sound-waves and many other periodic changes. The interaction of solar and lunar tides, resulting in priming and lagging, spring and neap tides, is best dealt with by this method.

§ 8. LOGARITHMS *

§ 8·1. TEACHING THE THEORY OF LOGARITHMS

The subject of the use of logarithms in practice has been dealt with in the *Report on Arithmetic* in § 14; it remains to deal here with the theory. Apart from the big question as to the construction of logarithms there is really only one thing to learn in the elementary bookwork of the subject, viz. that the statements

$$x = \log_a N$$

$$\text{and} \quad N = a^x$$

are equivalent, but it often takes a lot of learning.

In teaching it may be helpful to speak of '$a=10$ to the index for a'.

The difficulty probably has two causes:

(i) the confusion of mind normally caused by a new and as yet unfamiliar notation;

(ii) lack of certainty of knowledge of the laws of indices.

Drill is therefore necessary in the proofs of the fundamental laws which may be set out thus:

$$ab = 10^{\log a} . 10^{\log b} = 10^{\log a + \log b};$$

$$\therefore \text{ by the definition of a log, } \log ab = \log a + \log b.$$

But the person who does not easily realise that

$$a = 10^{\log a}$$

may prefer

$$x = \log a \text{ means } 10^x = a,$$

$$y = \log b \text{ means } 10^y = b;$$

$$\therefore 10^{x+y} = ab,$$

$$\text{which means } x + y = \log ab;$$

$$\therefore \log a + \log b = \log ab.$$

Similarly for the other laws.

Such questions as: express the equality $2^3 = 8$ in the language of logarithms; find $\log_3 81$, etc., are certainly useful. Possibly, but

* For further information on the construction of logarithm tables see articles in the *Mathematical Gazette* by Dr. Henderson (Dec. 1930) and W. C. Fletcher (Jan. 1931).

more doubtfully, such conundrums as: translate $10^{2\log_{10}N}$ into a simpler form, may be employed.

Practice in translating suitable formulae or equations into logarithmic form is essential, especially such equations as have the unknown among the indices.

It is easy (for examiners and others) to set certain questions in such a way as to snare those whose feet are not firmly planted on the rock of the fundamental principle; even such a question as 'what is the value of $10^{0\cdot13}$' will upset many who might probably evaluate $(2\cdot35)^{1/3}$ correctly. Whether it is better to regard the former question as one to be answered instanter or to regard it as a special case of x^n made difficult by its very simplicity is a point on which teachers will probably differ. We urge that habitual recourse to the general principle is better than cultivation of devices to deal with special cases.

The new language naturally increases the pupil's difficulty of expressing himself properly, and further difficulty is produced by the use of elliptical expressions such as 'to multiply you add', like the familiar 'two minuses make a plus'. Abuse of the sign of equality so familiar in the simplification of complex fractions is a further source of trouble.

While, in actually effecting evaluations, it is best (see *Report on the Teaching of Arithmetic*, § 14·3) to omit written explanations, it is necessary to give such drill in expression as will ensure thorough mastery of principle and conduce to clear writing when that is requisite. Thus it may be useful to employ such exercises as evaluate $\log 3 - \log 2$; $\log 3 \times \log 2$; $\log 3 \div \log 2$. The last is particularly important since it occurs in many compound interest questions, e.g. in the solution of the equation $500 \times (1\cdot04)^n = 750$; the fact that a logarithm has to be divided by a logarithm throws a boy off his balance and he is apt to make a fool of himself.

The use of negative characteristics and operations upon them ought to have been mastered in the earlier mechanical work, but some revision will probably be necessary. Boys must appreciate that $\bar{1}\cdot25$ and $-0\cdot75$ are equivalent and that negative mantissae are scarcely ever used; e.g. to evaluate $(0\cdot2)^{0\cdot3}$,

$$\begin{aligned}\log N &= \log(0\cdot2) \times 0\cdot3 = \bar{1}\cdot301 \times 0\cdot3\\ &= -0\cdot3 + 0\cdot0903 = -1 + \cdot7903 = \bar{1}\cdot7903,\end{aligned}$$

etc.

Changing from one base to another should be included, if only for the further exercise it gives on the fundamental principle, and to this it should be referred.

Thus, if $X = \log_a N$, and we require $\log_b N$, we argue that

$$\log_a N = X, \text{ i.e. } N = a^X.$$

Taking logs. of both sides to base b, we get

$$\begin{aligned} \log_b N &= X \log_b a \\ &= \log_a N \log_b a. \end{aligned}$$

Alternatively,

$$\begin{aligned} N &= a^X = (b^{\log_b a})^X \\ &= b^{X \log_b a}, \text{ etc.} \end{aligned}$$

§ 8·2. THE CALCULATION OF LOGARITHMS BY AN ELEMENTARY METHOD

The method of obtaining logarithms by calculating and plotting fractional powers of 10, e.g.

$$10^{\frac{1}{2}},\ 10^{\frac{1}{4}},\ 10^{\frac{1}{8}}, \ldots$$

is familiar and useful. It is easily possible by this method to get results correct to two significant figures, and the method is also valuable as illustrating the meaning of fractional indices. But there is another method which is much less laborious, which can be pushed to a higher degree of accuracy and which leads on naturally to the higher theory. This method is not so generally known as it should be and it may be useful to describe it here.

The fundamental fact as to the relation between numbers and their logarithms is that addition to the logarithm corresponds to multiplication by a factor, i.e. that, if logarithms proceed in arithmetic progression, numbers proceed in geometric. All that is necessary therefore for the construction of a system is to put side by side a set of numbers in G.P. and a set in A.P. with the condition that they must be so closely packed that interpolation is possible with sufficient accuracy.

It is further convenient that the logarithm of unity should be zero. We therefore start the G.P. with unity and the A.P. with 0.

If at the outset we made 10 correspond to 1 we should be driven back on to the troublesome method of extracting roots; so it is better to choose an arbitrary factor for the G.P. and make the necessary adjustment afterwards.

The factor must be near unity so that the numbers may be closely packed, and for ease in effecting the multiplications it is convenient to take such a factor as 1·1 or 1·05 or 1·02 or 1·01. The first of these will serve our purpose and will be found to give logarithms correct to 3 figures; the others give progressively better results but involve more work.

The first step, then, is to form the powers of 1·1 until we reach

or get beyond 10. It is sufficient to work to 7 places and to retain 5. So we get the table:

1·1	1
1·21	2
1·331	3
1·4641	4
1·61051	5
1·77156	6
1·94872	7
2·14359	8
2·35795	9
2·59374	10
2·85312	11
3·13843	12
3·45227	13
3·79750	14
4·17725	15
4·59497	16
5·05447	17
5·55992	18
6·11591	19
6·72750	20
7·40025	21
8·14027	22
8·95430	23
9·84973	24
10·83471	25

If we plot these numbers against the reference numbers of the right-hand column as abscissae, we get points on the familiar compound interest curve. If we join the points by straight lines, we get a very fair approximation to the curve itself—probably at least as good as we could draw by hand, for it is not even possible to plot the numbers to a third figure with any great accuracy. If we follow tentatively the corresponding process in calculation, we shall interpolate by proportional parts. If we do this we find for instance that the reference number for 10 is 24·15 and for the single figure numbers they are:

2	7·26
3	11·51
4	14·53
5	16·88
6	18·79
7	20·40½
8	21·81
9	23·05
10	24·15

The reference numbers form an A.P. coordinated with the G.P. and are of course logarithms to the base 1·1; but, since we want an A.P. such that the anti-number of 10 is unity, we must divide our reference numbers by 24·15.

The results to 3 figures are:

2	0·301
3	0·477
4	0·602
5	0·699
6	0·778
7	0·845
8	0·903
9	0·955

which are in fact the logarithms correct to 3 figures or very nearly so.

The logarithm of any other number can be found in the same way, i.e. by interpolation we find its place in the table and divide the resulting reference number by 24·15.

The error due to interpolation by proportional parts, i.e. by treating the stretches of the curve between the calculated points as straight, varies of course with the position inside the stretch; at the worst it amounts to 0·0125 in the reference number. Division by 24·15 therefore results in a maximum error of 5 in the 4th figure of the logarithm.

It is evident from consideration of the curve that all intermediate reference numbers are too small—the better value for that of 10 for instance is 24·153. The errors are not therefore cumulative, and they are reduced in the division; in fact in the logarithms an error so great as ·0005 is hard to find.

§ 9. MECHANICAL OPERATIONS IN ALGEBRA

In algebra, as a generalisation of arithmetic, the simple mechanical processes, the four rules, simplification of fractions, will have to be dealt with and put to use. To whatever degree the pupil acquires a conscious realisation of the parallelism of the procedure in algebra and arithmetic, and the reason for some differences in detail, his progress will be correspondingly accelerated.

The differences come under four heads :

(i) In algebra there are no 'carrying' figures; there is no parallel in algebra to $7 \times 9 = 63$.

In arithmetic 1s. 3d. − 5d. can be given as 10d., but in algebra $(a+3b)-5b$ does not simplify beyond $a-2b$.

(ii) Brackets are more widely used in algebra.

(iii) Algebra has a use for the sign − which does not occur in arithmetic. Such an operation as $(3a-2b)$ multiplied by $(2a-3b)$ is not found in arithmetic.

(iv) The notations $2a$ and $2\frac{1}{3}$ should be compared.

The parallelism should be frequently referred to; for instance, in addition

$$\begin{array}{r} 3a+2b+c \\ 5b+2c \\ 9a \quad\quad +6c \\ \hline 12a+7b+9c \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{rrr} £3 & 2 & 1 \\ 0 & 5 & 2 \\ 9 & 0 & 6 \\ \hline £12 & 7 & 9 \\ \hline \end{array}$$

bear a facial resemblance that gives a feeling of familiarity. It also tends to emphasise the point that $2b$ can no more be added to $3a$ than can 2s. be added to £3 so as to give a monomial sum.

Again, compare £3 + 2s. = 62s. or £3·1. This point would, however, be more effectively made if in the money sums the shillings were replaced by French francs and the pence by Italian lire, since the ratios of values would then be variable.

In the same way in multiplication

$$\begin{array}{r} 312 \\ 21 \\ \hline 624 \\ 312 \\ \hline 6552 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{r} 3x^2+\ \ x+2 \\ 2x+1 \\ \hline 6x^3+2x^2+4x \quad\quad \\ 3x^2+\ \ x+2 \\ \hline 6x^3+5x^2+5x+2 \\ \hline \end{array}$$

are strictly parallel; they are in fact the same sum if $x=10$. The substitution of 10 for x should not be regarded as a check but as something of greater significance. Further, examples involving different powers of x may serve to introduce the idea of degree.

In harder examples and particularly in division the differences mentioned in (i) to (iv) above will modify the completeness of the parallelism, but an occasional comparison with the arithmetical counterpart will be useful.

The pupil introduced to the identity $p(a+b) \equiv pa+pb$ may be reminded that it is a generalisation of the principle which he has been using when he has added 6 times 723 to 40 times 723 to find 46 times 723.

In fractions boys make many mistakes, especially in the use of cancellation, that they would not make in a similar arithmetic sum. They would make fewer of these mistakes if they acquired the conviction that the procedure with letters was essentially the same as they were already accustomed to use with numbers.

They may not like the look of $\frac{a}{b}+\frac{b}{d}=\frac{ad+b^2}{bd}$, but they will not be tempted to cancel b to make it look nicer or to write $\frac{a+b}{b+d}$ or to discard the denominator, if they realise it is the same kind of sum as $\frac{2}{3}+\frac{3}{5}=\frac{10+9}{15}$.

One caution is necessary in fractions, namely that, when a common denominator is used, the use of the lowest common denominator is sometimes practically essential.

Little harm is done if, in adding $\frac{1}{15}+\frac{1}{21}+\frac{1}{35}$, the denominator is taken as $15 \times 21 \times 35$; since the answer will be easily reduced to its lowest terms.

But in $\frac{1}{x^2-5x+6}+\frac{1}{x^2-4x+3}+\frac{1}{x^2-3x+2}$, the simplification is unmanageable if $(x^2-5x+6)(x^2-4x+3)(x^2-3x+2)$ is used as the common denominator.

Again, in fractional equations the use of any common denominator but the lowest as a multiplier will introduce a root or roots that are not valid. The lowest common denominator must be used.

Square root in algebra is on a different footing from the operations already mentioned.

In arithmetic a boy understands something of the explanation of these operations or at any rate they are explicable without recourse to algebra. And so the algebra is better understood by reference to the arithmetical knowledge already possessed. But in square root algebra sheds light on the arithmetical process. And, while it may be advisable that familiarity with arithmetical square root should

serve as an introduction to the algebraical, it is to algebra (or a corresponding geometrical diagram) that he must look for the *raison d'être* of the process. It shows how the divisor is contrived so as to give the right quotient, and it shows how the sum of all the subtrahends at any stage is a perfect square.

Incidentally it might be made an excuse for showing that the appropriate algebraical identity could be used to furnish an arithmetical process for finding any root if it were worth the trouble, but it should not be made an excuse for setting complicated examples in examinations.

§ 10. NEGATIVE NUMBERS. DIRECTED NUMBERS

The first time that a boy sees -3 standing by itself may very well be accidental. In working out some problem he may follow the usual practice of getting the x's to the left-hand side and may arrive at $-x=-3$ (or perhaps $-15x=-60$). This may perhaps, and certainly ought to, strike him as quite without meaning. What is the teacher to do?

It may be convenient to take the opportunity to start on the spot an account of 'directed numbers'. This is perhaps unlikely; the end of the period may be near or the general level of the class may suggest that a few weeks later would be better. In this case the teacher should say 'you will learn all about a meaning that can be given sometimes to -3 when we start on directed numbers the week after next. For the present, notice that if you had brought the x's to the right-hand side and the numbers to the left you would have got $3=x$ (or $45=15x$), which of course is the same as $x=3$, so that when you get $-x=-3$ another time all you need do is either "add $x+3$ to both sides", in which case you get $3=x$, or, if you prefer, change the sign of both sides, in which case you get $x=3$'. Such an accidental first introduction to negative numbers will very likely stimulate curiosity and interest for the lessons on directed numbers. In fact some teachers go out of their way to ensure that such an accident shall happen just before they are ready to begin the subject.

This beginning should not be unduly delayed, for the accidents mentioned above are likely to occur in the early months of the teaching of algebra. The general idea is picked up quickly, and it widens the possibilities for examples considerably. For a discussion of some of the subtleties the teacher is referred to the Appendix.

(·1) *Introduction of Directed Numbers.* The procedure is to explain the advantage of a number scale extending on both sides of zero from some example in which it is actually used. For instance, the thermometer gives excellent illustrations of what is meant by -3, $+3$ (when the $-$ and the $+$ are adjectives, not commands to add or subtract). Such expressions as $(-3)+(+7)$ or $(-3)-(+7)$ can also be explained by reference to rising and falling temperatures.

Of course other illustrations must be used almost at once; the familiar ones are: going up and down stairs; a bank account which can be overdrawn; distances along a road to E. or W. of a milestone; times which can be before or after noon; and, as a sort of

generalisation, anything which can be shown along a scale with a zero not at one end but in the middle.

What must be aimed at is variety of illustration and making the boys see that the same rules work in all cases. There must also be a good deal of drill, preferably in short periods of oral work, so that the boys will *know* the rules of sign for such things as $(-3)+(+7)$, $(-3)-(-7)$ .... It is a good thing at first to distinguish between the uses of $+$ and $-$ as adjectives and their (already well known) uses as commands to add or subtract, by reading $(+3)$ as 'positive three', (-3) as 'negative three', perhaps writing (-3) as $\bar{3}$.

(·2) *Illustration of Multiplication and Division.** For this it is necessary to find two quantities measured by directed numbers such that their product has a meaning. To do this take one of the quantities to be a time and the other to be any of the previous illustrations. For example, we may take $s=vt$ (distance = veloc. × time), which will illustrate all the rules of signs quite well.

Or, again, consider a man's attempt to save money.

(i) If a man saves £2 each month, 4 months from now ($+4$ from now) he will have saved £8. This gives a meaning to $(+2)\times(+4)$, showing that it is $(+8)$, i.e.

$$(+2)\times(+4)=(+8).$$

(ii) If a man saves £2 each month, 4 months ago (-4 from now) he was £8 worse off than now, i.e. $(+2)\times(-4)=(-8)$.

(iii) If a man loses £2 each month, 4 months from now ($+4$ from now) he will be £8 worse off than now, i.e.

$$(-2)\times(+4)=(-8).$$

(iv) If a man loses £2 each month, 4 months ago (-4 from now) he was £8 better off than he is now, i.e. $(-2)\times(-4)=(+8)$

In all four of these examples the sign $\times$ is a symbol of operation, but the signs $+$ and $-$ are symbols of directed numbers.

Again, a liquid now at $x°$ and being heated at $y°$ per minute will have after t minutes the temperature $x+yt$ degrees. Here y and t may be positive or negative, and the results obtained by the rule of signs will be found to agree with those obtained by common sense.

The use of the rules $(+a)(-b)=(-ab)$, etc., having been obtained from examples in this way, it should be pointed out how far they agree with the rules for ordinary signless numbers.

With signless numbers, when $a>b$ and $c>d$,

$$(a-b)(c-d)=ac-bc-ad+bd,$$

so that in the multiplication the terms $-b$ and $-d$ have actually led to $+bd$.

* Another illustration is given in Appendix A 9.

Also the examples show that with directed numbers it is convenient to call $(-b)\times(-d)$ or (negative b) $\times$ (negative d) equal to $+bd$. So this rule of algebra for the new numbers is the same as for the old, and any other instance will confirm that this is always the case. Plenty of drill in short periods of oral work is again indicated here.

There is a warning for the teacher at this point. He must not say that 'for signless numbers $(a-b)(c-d)=ac-bc-ad+bd$, *therefore* for directed numbers $(-b)\times(-d)=(+bd)$'. The *therefore* is wrong. There is no *logical* connection between the two statements, though there is a very close connection in the innate desire of the human mind to make a law which is applicable to one range of quantities apply also to another range of quantities. If the laws were not the same for the algebra of directed numbers as for the algebra of signless numbers, the unity of algebra would be lost and we should have two algebras with different apparatus.

When the work outlined above has been done the boy is ready for the ordinary elementary algebra in which the rules work equally well whether the numbers are positive or negative.

(·3) *Further Remarks.* It is a good plan, once the initial stages have been passed, to allow the distinction between 3-signless and 3-directed to sink into the background. The theory is full of pitfalls, and very few teachers are in a position to avoid error themselves.

An account of the whole matter both from the historical point of view and from the point of view of modern algebras is given in the Appendices A 1 and A 2, and some examples are given in Appendix A 3 of the difficulties and fallacies into which an unwary teacher may be led.

It is not too much to say that nine teachers out of ten would be unable to find the fallacy in the proof given there that the signless number 3 and the directed number $+3$ are the same. Till they are quite confident about such things themselves they should 'refrain, yea, even from good words'.

§ 11. ORAL WORK

Oral work in algebra as in arithmetic has a part to play, the value of which cannot be overestimated but is rarely appreciated to the full.

Questions for oral work may be framed to deal with one point of principle at a time, excluding extraneous difficulties.

Thus in the process of instilling the principles for solving simple equations, batches of questions of the several types (i) $x+3=7$, (ii) $x-2=9$, (iii) $2x=12$, (iv) $\frac{x}{2}=11$ ensure that the principles shall be grasped by giving the pupils an opportunity of translating the algebraical statements into simple problems, such as 'a number increased by 3 is 7, what is it?' This work takes less time than such written solutions as would guarantee that the principles are being applied and is more effective.

The impression a boy's mind takes of the formulation of a new principle is modified by the texture of his mind, by the narrow range of his vocabulary, and by his limited experience. When a class is engaged on a new piece of work, boys jump to conclusions some of which are right and some false. In *viva voce* work the teacher has the opportunity of marshalling facts so that the jump is more likely to be in the right than the wrong direction; he can deal with the wrong conclusions at once, and so conform the right ones that they receive conscious recognition and are distinguished from distorted substitutes. In *viva voce* work the teacher sees the boy's mind at work and can consequently influence his way of thinking in a way which is quite impossible with written work. Incidentally the class almost unconsciously enlarges its vocabulary and learns to use new words and processes correctly.

When the principles are grasped in this way and written work is set, the boy's attention is freed to deal with the proper lay-out of the solution and to apply the principles to more difficult examples.

Many examples ordinarily set as written exercises are much more suitable for oral work. This is especially the case in factorisation. It frequently happens that a correct answer is written down without full consideration, and it is of course pure luck that it is right, e.g. if a boy writes down $x^2-5x-6=(x+1)(x-6)$, there is no guarantee that he has checked for the coefficient of x, and some boys acquire a habit of ignoring this term and trusting to luck for the correctness of their answer. When they have answers available,

they use them as a substitute for the verification they should make for themselves, and are even ready to use the answers as a broad hint. If the work is done orally, they will be questioned on such a point and thus acquire the habit of being satisfied with their answers only after they have verified for the coefficient of x.

We give here a few further suggestions for oral work that do not appear in all text-books. The work should be entirely oral and the class need not make notes.

Before setting a class to do a group of problems, it is useful to discuss the method of procedure for several questions, e.g. 'What shall be taken for the unknown?' Express various quantities mentioned in the question in terms of that unknown. In other questions discuss the procedure in the same way.

Before drawing graphs, discuss (i) which quantity to measure across the page, (ii) what scales to take.

In simplification of fractions, what common denominator should be taken?

In fractional equations, by what shall we multiply both sides?

In simultaneous equations, which unknown is it easier to eliminate?

In simultaneous equations, for which letter shall we substitute?

In factors of the type $x(y+3)-5(y+3)$, what is the factor common to both terms?

When a particular mistake, e.g. in the rules of signs, is common in a batch of work, a set of quick oral questions may be designed to correct it, such as, what is $7-(7-1)$; $a-(a-b)$; $a-(-b)$? etc.

Good text-books contain exercises suitable for these purposes. The teacher should not use them perfunctorily, but with a full recognition of their intended usefulness, and even regard them as suggestive of others he can provide himself till their purpose is achieved. As 'oral work' they are intended for class discussion in introducing a new topic or correcting a prevalent error; but they may also be used for rapid revision, and in this case more difficult exercises may be set.

They may be supplemented by 'mental questions', that is, sets of questions to which answers are written down on paper without discussion and without any written work. Such questions will test the correct application rather than a sound understanding of a principle.

§ 12. PROBLEMS

Before putting down an equation involving x, the meaning of x should be clearly stated.

It is recommended that in all mathematics less advanced than the trigonometrical solution of triangles, letters should be used as abbreviations for numbers only, not for distances, weights, sums of money, or even ages. When in doubt it is well to take x as the number sought; but beginners often assume too light-heartedly that when x has been found it is necessarily the answer.

Example A. I can walk down a moving stairway in 30 seconds, taking 26 steps, or in 18 seconds, taking 34 steps. How many steps high is the stairway?

SOLUTION: The stairway moves down x steps per sec.
The first time I descend $26+30x$ steps.
The second time I descend $34+18x$ steps.
These are equal.

$$\therefore\ 26+30x=34+18x;$$
$$\therefore\ 12x=8;$$
$$\therefore\ x=\tfrac{2}{3};$$

$$\left.\begin{aligned} \therefore\ 26+30x&=26+20=46,\\ \text{and}\quad 34+18x&=34+12=46; \end{aligned}\right\}\ \therefore\ \underline{\textit{Height is } 46 \textit{ steps.}}$$

Tabulation often proves to be the shortest and clearest way of making the definitions and statements necessary before putting down the equation.

Example B. Three years ago Ted was four times as old as Eva: in seventeen years from now Eva's age will be $\frac{2}{3}$ of Ted's. How old are they now?

SOLUTION (Mentally. How many people? Two. How many dates? Three. Therefore six pigeon-holes):

	3 years ago	Now	17 years ahead	
Ted's age -	$4x$	$4x+3$	$4x+20$	years.
Eva's age -	x	$x+3$	$x+20$	

$$x+20=\tfrac{2}{3}(4x+20);$$
$$\therefore\ 3x+60=8x+40;$$
$$\therefore\ 20=5x;$$
$$\therefore\ x=4.\quad \left.\begin{array}{l}\underline{\textit{Ted is } 19}\\ \underline{\textit{Eva is } \ 7}\end{array}\right\}\ \underline{\textit{years old now.}}$$

Note.—A good general principle is not to dig facts unnecessarily soon out of the book while filling up the pigeon-holes, but to be left with one fact in reserve for making an equation when the tabulation is complete.

So many x, y problems can be done more efficiently by using one symbol only that a few types in which the use of two is certainly advantageous are worth noticing.

Example C. A two-figured number is multiplied by 6 by placing a 0 between its figures. The original number is less by 63 than that formed by interchanging its figures. Find it.

SOLUTION: Original number, written as 'xy' $= 10x+y$.

Second number, written as '$x0y$' $=100x+y$.

Reversed number, written as 'yx' $= 10y+x$.

$$100x+y=6(10x+y);\qquad \therefore\ 8x-y=0. \quad\text{..............(1)}$$
$$10y+x=10x+y+63;\qquad \therefore\ \underline{-x+y=7}. \quad\text{..............(2)}$$
$$\therefore\ 7x=7;$$
$$\therefore\ x=1;$$
$$\therefore\ y=8.$$

<u>*The number is* 18.</u>

Example D. A mule and a donkey were going to market laden with wheat. The mule said, 'If you gave me one measure, I should carry twice as much as you; but if I gave you one, we should bear equal burdens'. Tell me what were their burdens.

SOLUTION:

	At first	After 1st change	After 2nd change	
Mule carried -	x	$x+1$	$x-1$	measures.
Donkey carried -	y	$y-1$	$y+1$	

$$x+1=2(y-1); \quad \therefore\ 2y-x=3; \quad \dots\dots(1)$$

$$\text{and} \quad x-1=y+1; \quad \therefore\ -y+x=2; \quad \dots\dots(2)$$

$$\therefore\ y=5;$$

$$\therefore\ x=7;$$

$$\therefore\ \left.\begin{array}{l}\textit{Mule carried } 7\\ \textit{Donkey carried } 5\end{array}\right\}\ \textit{measures.}$$

Problems may serve to make the use of 'negative' or 'directed' numbers more convincing to beginners.

Example E. At a shooting-range you pay twopence for every miss and win a shilling for every hit. If after 60 shots you are poorer by 6s. 6d. than when you started, how many hits did you make ?

SOLUTION :

	Number	Pence gained by shooter	
		By one shot	By all shots
Hits -	x	12	$12x$
Misses -	y	-2	$-2y$

$$12x-2y=-78; \quad \therefore\ 6x-y=-39, \quad \dots\dots(1)$$

$$x+y=\ \ 60; \quad \dots\dots(2)$$

$$\therefore\ 7x\ \ =\ \ 21;$$

$$\therefore\ \ x\ \ =\ \ 3. \qquad \underline{\textit{He made 3 hits.}}$$

An alternative method of statement, not using tabulation :

Let x be his number of hits. By these he gained $12x$ pence.

Let y be his number of misses. By these he lost $2y$ pence.

Equation 1 (from numbers of shots) : $x+y=60$.

Equation 2 (from numbers of pence gained) : $12x-2y=-78$.

(The rest follows as above.)

Elementary problems most efficiently done by the use of three symbols are rare.

Example F. A, B and C can walk at 3, 4 and 5 m.p.h., and can bicycle at 12, 10 and 8 m.p.h. They have two bicycles. How can they most quickly travel 18 miles ?

SOLUTION :

	Miles	M.P.H.	Hours
A	Bi. x	12	$\frac{x}{12}$
	W. $18-x$	3	$6-\frac{x}{3}$
B	Bi. y	10	$\frac{y}{10}$
	W. $18-y$	4	$4{\cdot}5-\frac{y}{4}$
C	Bi. z	8	$\frac{z}{8}$
	W. $18-z$	5	$3{\cdot}6-\frac{z}{5}$

A's whole time $=6-\frac{x}{4}$ hours.
B's " " $=4{\cdot}5-\frac{3y}{20}$ "
C's " " $=3{\cdot}6-\frac{3z}{40}$ "

These must be equal, for time is wasted if one man arrives after the others.

$$\therefore\ 4{\cdot}5-\frac{3y}{20}=6-\frac{x}{4}, \quad\ldots\ldots(1)$$

$$\text{and}\quad 3{\cdot}6-\frac{3z}{40}=4{\cdot}5-\frac{3y}{20}; \quad\ldots\ldots(2)$$

$$\therefore\ 5x-3y=30, \quad\ldots\ldots(1)$$

$$\text{and}\quad 2y-z=12. \quad\ldots\ldots(2)$$

Also, because the two bicycles travel 18 miles each,

$$x+y+z=36; \quad\ldots\ldots(3)$$

$$\therefore\ 6x=78;$$

$$\therefore\ x=13,$$

$$y=11\tfrac{2}{3},$$

$$z=11\tfrac{1}{3};$$

$\therefore$ *A takes $2\frac{3}{4}$ hours,*
B takes $2\frac{3}{4}$ hours,
C takes $2\frac{3}{4}$ hours.

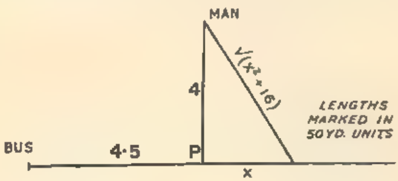
(To illustrate their journey graphically is of some interest. It is curious that they share the bicycles so nearly equally among them, though if they had only one bicycle they would ride it for 10, $6\frac{2}{3}$ and $1\frac{1}{3}$ miles.)

A clearly marked figure can often with advantage supersede much of the written or tabulated statement.

Problems leading to quadratic equations should include some in which both solutions are directly intelligible.

Example G. A man in an open field, 200 yards from P, the nearest point of a straight road, sees a bus 225 yards from P, and approaching P along the road. If he can run at two-thirds of the bus's rate, for what point on the road should he aim so as to catch it ?

SOLUTION :



he just catches the bus, $\sqrt{(x^2+16)}=\frac{2}{3}(4{\cdot}5+x)$

$$=3+\tfrac{2}{3}x\,;$$

$$\therefore\ x^2+16=9+4x+\tfrac{4}{9}x^2\,;$$

$$\therefore\ 5x^2-36x+63=0\,;$$

$$\therefore\ x=3 \text{ or } 4{\cdot}2.$$

He just catches the bus if he runs to a point 150 *or* 210 *yards from P*, and he catches it with something to spare if he runs to any point between these.

Many problems leading to quadratic equations, having a certain type of symmetry in the data, suggest a false interpretation for one of the roots of the equation.

Example H. The price of horses being increased by £6 each, five less are bought for £600 than formerly. Find the prices.

SOLUTION :

	Number	Cost of one (£)	Cost of all (£)
Cheap -	$\frac{600}{x}$	x	600
Dear -	$\frac{600}{x+6}$	$x+6$	600

$$\frac{600}{x}-\frac{600}{x+6}=5;$$

$$\therefore \frac{120}{x}-\frac{120}{x+6}=1;^*$$

$$\therefore 120x+720-120x=x^2+6x;$$

$$\therefore 0=x^2+6x-720$$

$$=(x+30)(x-24);$$

$$\therefore x=[-30 \text{ or}]\ 24;$$

$\therefore$ £24 *and* £30 *are the prices.*

Boys are very liable to deduce the answer £30 from the solution $x=-30$ instead of adding £6 to the cheap price given by the positive solution. This tendency should be remedied by a plentiful admixture of problems in which the rejected solution of the equation bears no resemblance to any part of the answer.

Example I. A runner in a mile race slows down by 2 feet per second after running 240 yards and quickens to his original rate again for the last 80 yards. The whole mile takes him 4 min. 48 sec. Find his two rates.

SOLUTION :

	Feet	Seconds	Ft. per sec.
Fast -	960	$\frac{960}{x+2}$	$x+2$
Slow -	4320	$\frac{4320}{x}$	x

$$\frac{960}{x+2}+\frac{4320}{x}=288;$$

$$\therefore \frac{10}{x+2}+\frac{45}{x}=3; \quad \text{(cf. footnote to Ex. } H)$$

$$\therefore 10x+45x+90=3x^2+6x;$$

$$\therefore 0=3x^2-49x-90$$

$$=(3x+5)(x-18);$$

$$\therefore x=[-1\tfrac{2}{3} \text{ or}]\ 18;$$

$\therefore$ 18, 20 *ft. per sec. are his rates.*

* Boys oftenmake their working unnecessarily heavy by omitting to divide through at this stage by any numerical factor which may be common to every term.

Many questions in mathematics illustrate the important principle, 'Don't rush needlessly into approximations. If you stick to exact quantities and short names as far as you can, the approximations and longer names may turn out not to be needed at all.' Problems are among the types of question which can exhibit this principle at work.

Example J. *A*, who can swim $1\frac{1}{2}$ m.p.h. in still water, swam two miles upstream and back. On another day, when the stream was flowing twice as fast, *B*, who swims 2 m.p.h. in still water, did the same swim in the same time. How long did each take ?

SOLUTION : Stream ran x m.p.h. on *A*'s day, $2x$ on *B*'s.

$$A\text{'s time} = \frac{2}{1\frac{1}{2}-x}+\frac{2}{1\frac{1}{2}+x}\text{ hrs.} \qquad B\text{'s time} = \frac{2}{2-2x}+\frac{2}{2+2x}\text{ hrs.}$$

$$= \frac{6}{2\frac{1}{4}-x^2} \quad ,, \qquad = \frac{1}{1-x}+\frac{1}{1+x} \quad ,,$$

$$= \frac{24}{9-4x^2} \quad ,, \qquad = \frac{2}{1-x^2} \quad ,,$$

$$\frac{24}{9-4x^2}=\frac{2}{1-x^2};$$

$$\therefore\ 12-12x^2=9-4x^2; \qquad \therefore\ A\text{'s time}=\frac{24}{9-1\frac{1}{2}}\text{ hrs.}=3\text{ hrs. }12\text{ min.}$$

$$\therefore\ x^2=\frac{3}{8}. \qquad B\text{'s time}=\frac{2}{1-\frac{3}{8}}\text{ hrs.}=3\text{ hrs. }12\text{ min.}$$

The most noteworthy point in this solution is that we avoid substituting an approximate square root of $\frac{3}{8}$ for x.

The more we attempt to apply elementary mathematics to real-life problems, the more we find that man does not live by equations alone. But even so, equations may give him a good start.

Example K. An electric battery of *AB* similar cells is made of *B* rows of *A* cells each, the rows being in parallel and the cells of each row in series. The electromotive force of such a battery is then *A* times that of one cell, and its internal resistance is *A*/*B* times that of one cell.

If any number of such batteries are placed in series in a circuit of external resistance *R*, the current varies as (sum of E.M.F.'s of batteries)÷(*R*+sum of internal resistances of batteries).

How would you arrange twenty similar cells of internal resistance ·2 ohm each so as to drive the maximum current through a circuit of external resistance ·36 ohm ?

SOLUTION: First suppose the cells arranged in one 'battery' as defined above.

$$\text{Then}\quad B=\frac{20}{A};$$

$$\therefore\ \text{Internal resistance}=\frac{A^2}{20}\times\cdot 2\text{ ohms}$$

$$=\frac{A^2}{100}\text{ ohms.}$$

$$\text{Current}\propto\frac{A}{\cdot 36+\dfrac{A^2}{100}},$$

$$\propto\frac{A}{A^2+36}.$$

To make this a maximum, $\dfrac{A^2+36}{A}$ must be a minimum.

Let $\dfrac{A^2+36}{A}=Z$; $\therefore\ A^2-AZ+36=0$;

$\therefore\ 2A=Z\pm\sqrt{Z^2-144}$;

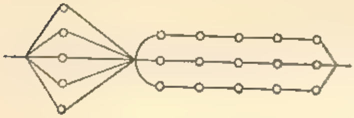
$\therefore$ Minimum $Z=12$, when $A=6$ and $B=3\frac{1}{3}$.

This suggests that three series of cells should be in parallel for the greater part of the way, and four for the rest of it, thus:

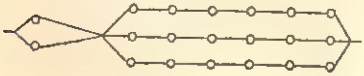
This arrangement makes $\dfrac{\text{sum of E.M.F.'s of batteries}}{R+\text{sum of internal resistances}}$ equal to 8·26 times a constant depending on the cell; and this 8·26 turns out, on examining a few other likely-looking arrangements, to be the highest possible value.

The three closest competitors are these:

which gives 8·20



which gives 8·18



which gives 8·14

§ 13. SIMPLE SERIES

It has been customary to deal with two kinds of series, arithmetical and geometrical progressions, in the elementary algebra course. As Professor Whitehead has pointed out, in his *Introduction to Mathematics* (Home Univ. Libr.), this is a narrow and inappropriate way in which to introduce the great subject of Series.

Elementary work will be concerned almost entirely with progressions; but this is no reason for starting with arithmetic and geometric progressions and neglecting the main ideas about series in general. It is the essence of a series in our sense of the word to possess an nth, or general, term, and a good introductory exercise is to ask questions like the following:

If the series	1, 2, 3, 4 ...	$1^2, 2^3, 3^4 \ldots$
	1, 3, 5, 7 ...	$3^{-1}, 5^{+2}, 7^{-3} \ldots$
	1, 4, 9, 16 ...	$1, -2, 3, -4 \ldots$
	2, 5, 10, 17 ...	3, 6, 12, 24 ...

continue in the way suggested by the first few terms, what are their nth terms? If the general terms are

$$3n+4,\ (1+n^{-1})^n,\ \log\left(\frac{n+1}{n}\right)\ldots,$$

give the first few terms.

Questions demanding skill and thought can be included in this introductory work, and they form a better exercise than the mere summation of arithmetic progressions. The idea of summation is quite outside the original idea of series, and it is wrong to allow boys to think that the only thing that ever has to be done to a series is to sum it. We thus follow Prof. Whitehead in recommending that an important modification should be made in the traditional manner of introducing this part of the subject.

When the inevitable summation of arithmetic progressions is reached, it will be found that the boys who have been brought up in well-conducted nurseries have already summed the number of pips in the ace, 2, 3, 4 ... 9, 10 of, say, diamonds, by putting the 10, 9, 8 ... 3, 2, and ace of, say, spades underneath them.

The fundamental rule for the summation of arithmetic progressions is, 'put down the series forwards and backwards and add'. The dull boy will need some considerable practice at carrying out the operation; before starting it seriously he should have had

practice at answering questions arising out of the general term as suggested above; and he should not begin by carrying out substitutions in a formula. The learning of the formula can be left till a much later stage.

It is most important to make it clear why it is only arithmetic progressions that can be summed by the process of writing down the series forwards and backwards.

In summing geometric progressions, it is only necessary to teach boys to write down the series twice, once in its original form and again multiplied by the common ratio, and to subtract, thus:

$$x = 1 + 2 + 4 + 8 + 16,$$
$$2x = \quad 2 + 4 + 8 + 16 + 32,$$
and subtracting $$x = -1 + 32 = 31.$$

The first example should be one like that given, in which all the terms can be written down without much trouble, and in which the result is easily checked by addition.

A second method should also be taught which will provide an opportunity for a revision of 'division', which is the more necessary nowadays when less time is spent in doing such routine work. The method may be indicated roughly thus:

Divide $1-x^2$ by $1-x$. Divide $1-x^4$ by $1-x$. Divide $1-x^5$ by $1-x$.

What is the result of dividing $1-x^{17}$ by $1-x$?

Hence $1+x+x^2+x^3+x^4+\ldots+x^{n-1}=(1-x^n)\div(1-x)$. Thus the factors of $1-x^n$ and the formula for summing a geometric progression have been made obvious rather than taught as formulae.

If infinite geometric series are included, it is essential to avoid inaccurate statements and equally essential to put things in a way suited to the capacity and immaturity of the class. All that is necessary may be attained by careful examination of the simplest cases. Thus it is evident that the sum $1+2+4+\ldots$ gets more and more unmanageable as we take more terms; each term in fact is greater than the sum of all the preceding terms.

On the other hand, it is easy to see, apart from any formula, that the sum $1+\frac{1}{2}+\frac{1}{4}+\ldots$

(*a*) never gets beyond 2;

(*b*) never reaches 2;

(*c*) may be made as near to 2 as we please.

For wherever we stop we are short of 2 by precisely the term last added.

All the three characteristics (*a*), (*b*), (*c*) of this series can be made still more evident by a simple geometrical illustration. A line AZ, two inches long, is drawn and lengths $AB=1$ inch, $BC=\frac{1}{2}$ inch, $CD=\frac{1}{4}$ inch and so on are marked off in succession. It is evident

that the addition of each extra length brings the final point nearer and nearer to Z, that the point can be brought *as near as we please* to Z, but that Z can never be reached.

The fact that in this instance we can keep the limit in sight, the number 2 if we take numbers only, or the length 2 inches if we draw the line, makes the argument easy; the argument cannot be so simply stated for $1+\frac{1}{2!}+\frac{1}{3!}+\ldots$ because we cannot predict and indeed cannot even state, the number required.

It is a little less easy, but still not difficult, to deal similarly with the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$. Here we are not likely to guess the number required, but can obtain it by a tentative use of the formula, viz. $\frac{2}{3}$. Starting at 1 to get to $\frac{2}{3}$ we ought to subtract $\frac{1}{3}$; we actually subtract $\frac{1}{2}$, i.e. $\frac{1}{6}$ too much; we correct by adding $\frac{1}{4}$, i.e. $\frac{1}{12}$ too much, and so on. So now, starting with the $\frac{2}{3}$ in our mind and proceeding step by step, we get:

(*a*) alternately beyond and below $\frac{2}{3}$;
(*b*) never reach $\frac{2}{3}$;
(*c*) get as near to it as we please.

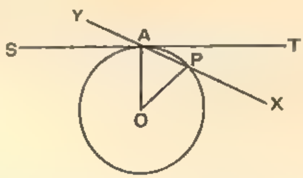
A geometrical illustration of the alternating approach to this limit $\frac{2}{3}$ is again illuminating and should be shown on the blackboard.

For further illustration of the new important ideas involved, it is desirable to take at the same time, or in close connection, the other case of a limit which naturally comes into elementary work, viz. the tangent to a circle. Here again ease and clearness depend on being able to lay down at the outset the critical line. Thus:

Let OA be a radius of a circle, SAT the perpendicular to OA, $YAPX$ any secant through A. The triangle OAP being isosceles, the angle OAP is acute, being less than 90° by $\frac{1}{2}AOP$.

Therefore:

(*a*) as P approaches A, the secant approaches SAT;
(*b*) so long as YX remains a secant it never coincides with SAT, but
(*c*) it may be brought as near to coincidence with SAT as we please.



To say 'the secant *becomes* the tangent for the perpendicular' is misleading, since a secant never is a tangent; what should be stated

is that, as the line YAX revolves round A, there is one and only one position in which it is not a secant: this exceptional position is the tangent.

Having once, so to say, stumbled on the fact that there are series such that for practical purposes their sums to any considerable number * of terms may be represented by some simple number, much as for practical purposes $\sqrt{2}$ may be taken as 1·414 or π as 3·1416, etc., it is desirable to examine some other simple series by straightforward arithmetic, i.e. to try to evaluate to a few places of decimals some series like

$$1+\frac{1}{2!}+\frac{1}{3!}+\ldots; \qquad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots; \qquad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

and even (using logarithms) $1-\pi^2/3!+\pi^4/5!-\ldots$.

And in any case the problem ought to be dealt with: How many terms of $1+\frac{1}{2}+\frac{1}{4}+\ldots$ must be taken to get beyond, e.g., 1·9999?

* Familiarity with the expression 'sum to infinity' or the more modern and in some ways preferable expression 'limiting sum' may be necessary for examination purposes. It is probably better to avoid using either expression at first and to refer to the 'limit of the sum'. The conventional terms can be introduced later, with the caution that they do not really denote a 'sum' at all.

§ 14. SYMMETRY

In geometry, symmetry of form is a comparatively rudimentary notion. It is easily perceived, and its characteristic properties easily formulated.

Its counterpart exists in algebra in such expressions as

$$2x^2+5xy+2y^2 \quad \text{and} \quad 2x^4-7x^3+10x^2-7x+2.$$

It is more strikingly shown by writing down the detached coefficients $2, 5, 2$ and $2, -7, 10, -7, 2$.

Powers of symmetrical functions are symmetrical in the same way, e.g.

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3,$$
$$(x^2-2x+1)^2=x^4-4x^3+6x^2-4x+1.$$

If a symmetrical expression is factorised, then either the factors are each symmetrical or they are symmetrical with respect to one another, thus :

$$2x^4-7x^3+10x^2-7x+2=(2x^2-3x+2)(x^2-2x+1),$$
$$2x^2+5xy+2y^2=(2x+y)(x+2y).$$

Other forms of symmetry have few applications in the elementary course, but that type in which an interchange of two letters leaves the expression unchanged has applications at an early stage.

$x^2-x(a+b)+ab$ remains unchanged if a and b are interchanged ; so does $a^3+a^2b+ab^2+b^3$.

The factors of the latter $(a+b)(a^2+b^2)$ exhibit the same property independently. The factors of the former $(x-a)(x-b)$ are interchangeable, when a and b are interchanged, and so form a symmetrical pair.

Geometry and trigonometry furnish many applications of this form of symmetry.

In $\triangle ABC$ if $AB=AC$ then $\angle B=\angle C$.

If BI and CI bisect angles B and C, then $\angle BIC$ is a symmetrical function of B and C which equals $A+\frac{1}{2}(B+C)$, or $90^\circ+\frac{1}{2}A$. In this function A is not involved in the same way as B and C.

$\sin(A+B)$ is a symmetrical function of A and B, and hence its expanded form, $\sin A\cos B+\cos A\sin B$, must be symmetrical.

Some expressions are symmetrical functions of three or more letters. They are unchanged by the interchange of any two of the letters.

$(x+a)(x+b)(x+c)$ is such a function ;

it equals $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$, the coefficients exhibiting the same sort of symmetry.

$(a+b+c)^2=(a^2+b^2+c^2)+2(bc+ca+ab)$, the sum of two terms each symmetrical in a, b and c.

$a^3+b^3+c^3-3abc$ factorises as

$$(a+b+c)\{(a^2+b^2+c^2)-(bc+ca+ab)\},$$

each factor being symmetrical in a, b, c.

$$b^2c+bc^2+c^2a+ca^2+a^2b+ab^2+2abc$$

factorises as $$(b+c)(c+a)(a+b);$$

here no factor separately exhibits the property, but the three form a symmetrical group.

Any measurement in a triangle which depends equally on all the sides will be represented by a symmetrical function of all the sides, thus:

$$\Delta=\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

s being $\frac{1}{2}(a+b+c)$ is itself symmetrical in a, b and c, and the other three factors form a symmetrical group.

If R is the circumradius and the point O the circumcentre of the $\triangle ABC$, $R=OA=OB=OC$ (A, B and C are interchangeable). If it is proved that $R=a/(2\sin A)$, it follows that

$$R=b/(2\sin B)=c/(2\sin C).$$

The notions of symmetry can be combined with those of degree in the discussion of formulae and in giving a sense of control in trigonometrical as in algebraical work.

The teacher is unlikely to find in elementary text-books sets of examples devoted to these notions. But when once the germ has been planted in the pupil's mind every lesson will provide material to stimulate its growth. Familiarity is acquired not by a condensed and isolated lesson, but by taking the opportunities afforded by the whole course of study to make frequent reference to symmetry.

§ 15. TRANSFORMATION OF FORMULAE

Exercises on the transformation or the change of subject of a formula are of great value in the teaching of elementary algebra. A boy who can transform such formulae as

$$t=\sqrt{\frac{Ph}{16(P-W)}} \quad \text{or} \quad \frac{1}{R}=\frac{CL}{t}\frac{V}{V-V_1},$$

to obtain any assigned letter has a competent knowledge of the use of formulae, of simplifying expressions, of an elementary and fundamental type of factor, and of the solution of simple literal equations.

The secret of successful teaching of this work lies in careful grading of examples so that each difficulty is mastered before the next is presented. This implies definite drill-work, and a definite setting-out of solutions.

The first transformations should be those of formulae containing only two letters, and these formulae should be either thoroughly familiar or easily appreciated. An easy example is $S=2n-4$ (angle sum of a polygon); a harder example is $C=\frac{5}{9}(f-32)$, for converting temperatures.

One or two numerical illustrations (e.g. 'if $C=100$, what is f?') may be worked before the general transformation is attempted. It is helpful to set out the general transformation line for line parallel with one of the preliminary numerical illustrations.

From the outset it must be emphasised that the letter required must appear *isolated* on one side of the final statement. Boys must be taught to apply this simple test to each example, and to realise that the fundamentally bad mistake, which is very common, is to show up such an answer as

$$r=\frac{A}{\pi r} \quad \text{for} \quad r=\sqrt{\frac{A}{\pi}}.$$

The formulae for the mensuration of a cylinder, i.e. $C=\pi d$, $C=2\pi r$, $A=\pi r^2$, $S=2\pi rh$, $T=2\pi r(r+h)$, form a convenient group whose transformations are of increasing difficulty.

The last two formulae of this group contain three variables and give rise to several transformations. It is of interest to perform all possible transformations of any given formula as soon as the class appreciates the purpose and processes involved.

After obtaining $r=\sqrt{(A/\pi)}$ from $A=\pi r^2$ it is natural to deal with the transformation of a formula in which the square root sign

occurs explicitly, e.g. $D=\sqrt{(\frac{3}{2}h)}$ for the distance of the horizon, or $T=2\pi\sqrt{(l/g)}$ for the time of swing of a pendulum.

The last illustration shows that the first steps in transforming a formula are, roughly, to clear our formula of root signs and of fractions. Then all the terms containing the letter required must be grouped on one side and factorised.

(A good example illustrating the second point here is: 'If $A=p+\frac{prt}{100}$, obtain a formula for p'.)

Some drill-work on this step is to write down a formula cleared of fractions, e.g. $2A=ha+hb$, or $a+amx=m-x$, and to set the class to derive formulae for as many letters as possible.

The standard setting-out of a transformation should now be built up.

Example. 'If $R=\frac{ns}{1+(n+1)s}$, find n.'

(i) *Clear of fractions and brackets.*

$$R[1+(n+1)s]=ns;$$
$$\therefore\ R+R(n+1)s=ns;$$
$$\therefore\ R+Rns+Rs=ns.$$

(ii) *Isolate on one side terms in n.*

Before collecting the terms it is a sound precaution to insist that all terms containing the letter required should be underlined so:

$$R+\underline{Rns}+Rs=\underline{ns},$$

and terms should be collected on the more convenient side to avoid a multitude of negative signs in the final quotient. Here we collect terms in n on the r.h.s., obtaining

$$R+Rs=ns-Rns.$$

(iii) *Express r.h.s. as 'n times something'.*

Since the letter required is a factor of every term on the r.h.s. it is a factor of the complete r.h.s. This should be explicitly pointed out.

$$\therefore\ R+Rs=n(s-Rs);$$
$$\therefore\ \frac{R+Rs}{s-Rs}=n,$$
$$\text{i.e. } \frac{R(1+s)}{s(1-R)}=n,$$

and so, finally, $n=\ldots.$

The numerator and denominator of the final formula should be factorised if possible.

Some such setting-out as this should be written down. When the wording of the statements is fixed in the memory the setting-out may be abbreviated.

Later in the course it is useful to point out that with suitable formulae there is an alternative setting-out in which the steps are planned by working backwards,

e.g. 'If $\frac{1}{f}=(u-1)\left(\frac{1}{a}+\frac{1}{b}\right)$ and we require b', we argue:

By looking at the formula we could isolate b, by having in the preceding line the value of $\frac{1}{b}$, and in the line preceding $\frac{1}{b}$, the value of $\frac{1}{a}+\frac{1}{b}$. Hence we write down

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{f(u-1)};$$

$$\therefore\ \frac{1}{b}=\frac{1}{f(u-1)}-\frac{1}{a};$$

$$\therefore\ b=\frac{1}{\frac{1}{f(u-1)}-\frac{1}{a}}.$$

Boys occasionally develop this type of solution for themselves.

Excellent manipulative practice arises when alternative solutions give apparently different answers and these answers have to be proved identical.

It will be pointed out that many formulae can be transformed only for some of the letters they contain,

e.g. $V=\pi r^2\left(h-\frac{r}{3}\right)$ will give h but not r.

In blackboard work and elsewhere it is wise to avoid in the same formula letters which easily appear alike in writing or in sound, e.g. u and v, b and l, m and n.

§ 16. VARIATION

§ 16·1. IDEAS UNDERLYING VARIATION

Arrival at the chapter styled 'Variation' brings an opportunity for revision, and for bringing into full consciousness ideas for which the materials have been long accumulating.

The phrases 'One quantity bears a constant ratio to another', 'One quantity is proportional to another', 'One quantity varies as another' are identical in meaning except that the first has a narrower field of application than the others, being applicable only to quantities of the same kind.

The feature, more or less new, in this terminology, to which special attention is necessary, is the use of the term 'quantity' and the consequent permissibility of speaking of two 'quantities' as proportional.

Hitherto in algebra each letter has generally been felt to mean a specific number, known or unknown, and a 'proportion' can only exist between four such numbers; the frequently underlying notion of a quantity or number which may take many values has not been absent, but it has received little or no explicit recognition. It will have come most nearly to the surface in the study of graphs or of mensuration formulae.

In this connection the word 'quantity' means anything measurable as distinct from quality which is not measurable and from the individual or thing in itself which may have a complex of quantities and qualities.

Thus a tree, a man, the moon are not themselves quantities, but each has many quantities as well as qualities; the height, age, weight of a tree or of a man; the diameter, surface, density of the moon are all 'quantities', each being a measurable entity; the colour of a tree, the character of a man are 'qualities' not susceptible of measurement in the ordinary sense.

Most of the 'quantities' with which we are concerned are subject to change or variation, and the business of science is to observe and so far as possible to correlate the variations of different quantities. In the simple case in which we are dealing with two quantities only, graphic representation gives us a picture of the numerical relations between them, and may afford a useful clue—in some cases more

than a clue—to a general expression for that relation, or may suggest that there is no relationship.

So far, the word 'variation' has no technical significance; it is merely a Latin equivalent for 'change'. To say that one quantity varies with another does perhaps imply some sort of connection; it is quite natural to say that a man's height varies with his age or that his weight varies with his height, not so natural to say that his height varies with his girth. In any case these are merely loose general statements with no precise mathematical signification, and similar statements can legitimately be made about 'qualities', i.e. about non-measurable entities: his health varies with the weather.

The technical use of the word belongs to its combination with the particular conjunction 'as'. 'The area of a circle varies with the radius' is a true but vague statement; 'the area varies *as* the square of the radius' is precise.

It conduces to clearness to adopt a device of notation. Thus W stands for weight; w lb. is the weight of, say, a heap of sand, while w_1 lb., w_2 lb. denote the weights of particular heaps of sand.

There are two methods of handling these questions, and they are not merely alternatives; one or other of them may be more convenient in a particular case.

The primitive method of expressing the relation $A \propto B$ (where A and B are names, not numbers) is by the equation

$$\frac{a_1}{a_2} = \frac{b_1}{b_2},$$

where $(a_1 b_1)$, $(a_2 b_2)$ are two pairs of corresponding measures of A and B.

Here each side of the equation is a true ratio, and it *does not matter what units are used* for the measurement of A and B respectively; e.g. 'the value of a bale of cotton presumably varies as the price per lb.'—the one is conveniently expressed in £, the other in pence; there is no need to express them in the same unit; or, again, 'the weight of a copper sphere varies as the cube of the radius'—here the two quantities cannot be expressed in terms of the same unit, and it does not matter whether the weight is expressed in pounds or tons or the radius in inches or feet.

On the other hand, the actual value of the ratio appearing in this method has no particular significance; it depends on the particular pair of cases shown; if we had taken $\frac{a_3}{a_4} = \frac{b_3}{b_4}$, it is improbable that $\frac{a_1}{a_2} = \frac{a_3}{a_4}$, and if it happened to be so, the fact is irrelevant.

The other method introduces a new and valuable principle, affording an illustration of the remarkable truth that the algebraic machine tends to create ideas.

Having written the equation $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ we may proceed, making use of the freedom given by algebra, to transform it to $\frac{a_1}{b_1}=\frac{a_2}{b_2}$.

In the general case in which A and B are quantities of different kinds, we can no longer retranslate this into the language of ratios, for A has no ratio to B. But now, not only is

$$\frac{a_1}{b_1}=\frac{a_2}{b_2} \quad \text{and} \quad \frac{a_3}{b_3}=\frac{a_4}{b_4},$$

but these expressions are all numerically equal, and we say that any numerical measure of A is k times the corresponding measure of B and may write $a=kb$, or where no confusion is likely to arise $A=kB$.

From this point of view k is primarily a mere number which remains constant throughout the question but has in itself no particular significance and is used simply to facilitate the mechanical work.

But certainly in very many cases which come under discussion, if not in all, it appears that k is a particular numerical value of some third 'quantity' intimately related with A and B. Thus if A is a mass and B a volume, k is the measure of a density; if A is money and B a quantity of material, k is a price; if A is distance and B time, k is a velocity, and so on.

The question of units may now become of importance. We may, of course, measure A in any unit we please, and B also in any unit, but the value of k will depend upon the particular units chosen. If A and B are quantities of the same kind, k is a true ratio, and if A and B are measured in terms of the same unit, k has a certain standard value, e.g. if A is the area of a circle (*N.B.*—'A circle' here means not a particular circle but any circle whatever) and B the area of the square on the radius and both are measured in the same units, k is π.

Otherwise, except in so far as k is used as a mere arithmetical device, any reference to its value will have to include a statement of the units of A and B: a density of so many pounds per cubic foot, or of grams per c.c., etc.

A good illustration of the advantages gained by this second method is afforded by many of Newton's propositions in the *Principia*, e.g.:

1. LXXI, in which he proves that the attraction of a sphere on an external point is the same as it would be if the whole mass were collected at the centre. He takes two spheres and works entirely by the method of ratios, using neither algebra nor trigonometry; the consequence is that the exposition seems prolix and obscure, while the same argument expressed in modern form is simplicity itself.

§ 16·2. THE TEACHING OF 'VARIATION'

The first point in teaching the subject is to revise and consolidate existing ideas of proportionality and to make familiar the expression of the relationship in both forms: a quantity A is proportional to a quantity B; A varies as B.

This involves bringing into clear consciousness the use of the general term 'quantity' (or 'magnitude') and the habit of thinking of *two* 'quantities' as proportional.

The common-sense test should be applied: if you double one, do you double the other? etc.

Before the relationship can be applied to the solution of a problem it must be expressed in algebraic form. It is in haste or carelessness at this point that the chief danger lies. Consider, for instance, a typical question:

The volume of a pyramid varies jointly as its height and the area of its base, and when the area of the base is 60 sq. ft. and the height is 13 ft. the volume is 260 cu. ft. What is the area of the base of a pyramid whose volume is 390 cu. ft. and whose height is 26 ft.?

Here, as so often, the first difficulty is to read this long statement and to translate it into the brief symbolism of algebra; one reads the first sentence and writes $V \propto h \,.\, A$.

The temptation is to go on at once to the second sentence and write $280 \propto 14 \times 60$, which of course is nonsense.

One has to pull oneself up and translate the statement $V \propto h \,.\, A$ into an equation;

either $$\frac{V_1}{V_2} = \frac{h_1 A_1}{h_2 A_2} \quad \text{or} \quad V = k \,.\, h \,.\, A.$$

Only then does the substitution of the particular values become possible.

When one or other of those equations has been written down there remain merely the familiar difficulties of substitution, manipulation and evaluation.

In a straightforward 'one step' question like the above there is perhaps little to choose between the two methods. An occasional question crops up—perhaps constructed for the purpose—in which ratios only are given; no actual values; the first method will then be the most rational and the easiest.

But there is a much larger class of questions in which the formation of a general expression for a 'quantity' is essential, this being only the first step to subsequent operations; the superiority, indeed the necessity, of the second method is then obvious.

The conclusion is that in the teaching of 'variation', even in the narrow sense of the word, both methods are essential, but that the second (the k-method) is the more important.

The law of 'joint variation' should not be neglected, though again it is merely a new way of looking at familiar facts—as should, of course, be made plain.

If the question be put in abstract form 'if A varies as B when C is constant, and as C when B is constant, what happens when B and C both vary?' it will appear puzzling—and the formal proof will be found difficult. But this very fact shows that treatment is necessary—but it must be treatment suited to the class, not the abstract treatment of the mathematician.

The question is simply that involved in all cases of 'compound proportion' whether worked by the method of reduction to unity or by that of ratios.

The distance described by a point moving with uniform velocity varies as the time if the velocity is constant and as the velocity if the time is constant: when both vary it varies *as the product*. For if you treble the time without changing the velocity you treble the distance, and if you *also* double the velocity you also double the distance which has been already trebled; i.e. you increase it six-fold.

This law of multiplication, not addition, is as important and fundamental here as it is in the case of 'combinations and permutations', and neglect is as fatal in the one case as in the other.

§ 16·3. FUNCTIONALITY

The habit of writing down the expression (e.g. $k \,.\, hA^2$) for the dependent quantity leads naturally to a wide and most important extension. So far the expression in question has consisted of a single term with its single constant; for even if two or more constants enter this joint variation, the two combine into one.

But questions naturally crop up leading to expressions consisting of several terms each with its own constant, and we are fairly embarked on the general problem common to physical science and to mathematics of investigating the connections between quantities when these no longer take the simple form of proportionality but may be indefinitely complex.

This problem has two aspects peculiar in their extreme forms to the pure mathematician and to the physicist or applied mathematician.

The former takes expressions which he studies for their own sake; the latter starts with a mass of observed data and attempts to find expressions to fit them. The former proceeds from the expression to the graph; the latter has the generally more difficult problem of discovering an expression to fit the graph.

In school practice it is of course natural to concentrate attention on those expressions and graphs which are likely to occur within

the range of school experience—whether in mathematics or science.

These will include at least kx, k/x, kx^2, $k/(ax+b)$, and for those into whose work they come $k \sin x$, $a \sin x + b \cos x$, $\log x$, a^x.

The direct process of proceeding from the formula to the graph will be familiar ; the reverse process of determining the expression from the graph, and in particular of determining the constants, will need more specific attention.

§17. ALGEBRA BEYOND THE 'ELEMENTARY' STAGE

Though some suggestions have been made for higher work, this report has mainly dealt with the work for 'elementary mathematics' of school certificate or matriculation examinations.

Leaving out of account mathematical specialists and other post-certificate pupils, such as science specialists, for whom mathematics is an important subsidiary subject, schools may be concerned with two stages of work.

A considerable number of the better school certificate candidates are able to devote much of their time to work which is outside the syllabus of elementary mathematics. Some of them will be working for an examination subject known as 'Additional Mathematics' or 'More Advanced Mathematics' and then their requirements will partly depend on the particular examination they take. Others will work at the subject of additional mathematics without being much concerned with examinations. The best course for such pupils is discussed in (17·1).

There will also be, at some schools, pupils who continue to learn mathematics after the certificate stage but definitely as a subsidiary subject of secondary importance. Their needs are dealt with in (17·2).

With mathematical specialists, this report is not concerned.

§17·1. ADDITIONAL MATHEMATICS

If the abler certificate candidates have a year or eighteen months in which to study less elementary work, the subjects taken up will usually be further Trigonometry, with Geometry and Algebra, and a selection from Calculus, Coordinate Geometry and Mechanics.

This Committee is firmly of opinion that nearly all the available time should be devoted to the new subjects and very little indeed to algebra. Perhaps two-thirds of the time should be given to mechanics, calculus, coordinate geometry, and much of the remaining one-third to trigonometry.

This leaves but little time for algebra. It has become customary to include in the examinations in additional mathematics some further snippets of algebra which are just outside the 'elementary' syllabus. This work might include :

(i) *Series* (if not included in elementary mathematics). Work with series such as A.P.'s and G.P.'s is both simple and stimulating,

provided that it is not allowed to become mere drill in the formulae concerned.

(ii) *Remainder Theorem.* This is sometimes included in elementary mathematics and, even if it is not, boys should usually be introduced to it in an informal way. In this more advanced stage it may be expected that the proof (using the ideas of identities) can be appreciated. Also some work with identities and undetermined coefficients will pave the way for 'partial fractions' later.

(iii) *Binomial Theorem.* It is easy to give a preliminary treatment of the binomial theorem for a positive integral index without a formal proof and without digressions into problems connected with ${}_nC_r$. It may be best to postpone the ${}_nC_r$ notation.

Some ideas connected with the more general theorem can well be given, e.g. it can be shown that $1+nx$ is an approximation to $(1+x)^n$ when x is small, or that $1+x$, $1+x+x^2$, $1+x+x^2+x^3$, ... are successive approximations to $1/(1-x)$.

(iv) *Theory of Quadratic Equations.* This will be a necessity if boys have made much progress in coordinate geometry. Heavy work, such as the calculation of $\alpha^7+\beta^7$, should be avoided, and the main part of the work should be concerned with the nature of the roots and in acquiring capacity to give the sum and product of the roots.

(v) *Certain topics of the elementary syllabus* from a slightly more advanced standpoint. For example, questions about logarithms involving more grasp of the theory, the more formal part of the Remainder Theorem (e.g. factorisation of symmetric expressions in a, b, c), a few new types of equations and graphs of the type referred to in § 17·2 (vii) below.

These things do not demand a great deal of time from the teacher, but their mention in examination syllabuses has the unfortunate effect that the examiner feels bound to set them and has to ring the changes on the limited types of questions available. It would probably be better if examiners were limited to slightly wider interpretations of the elementary syllabus such as are suggested in (v).

§ 17·2. POST-CERTIFICATE WORK

It is not suggested that algebra should be prominent; there are many interesting subjects available. The suggestions which follow are given on the assumption that, for one reason or another, algebra has been included in the course.

Some of the boys who take the course will probably be of mediocre ability, and such boys are not best taught by being taken over, say, the first quarter of the course followed by mathematical specialists. This must be borne in mind if the course is to be made interesting; it must be remembered that certain types of questions demand the

insight of the real mathematician, and these must be avoided. Suitable subjects for inclusion in the post-certificate course are :

(i) *Binomial Theorem for a positive integral index.* A more complete course than that suggested for Additional Mathematics. More drill will be needed if the boys are of low ability. But the formal work should be limited.

A digression into some simple numerical questions of permutations, combinations, and probability might prove interesting.

(ii) *Series to n terms*, with general terms not more complicated than, say, $A+Bn+Cn^2+Dn^3$.

(iii) *Infinite Series.* The merest elements of convergence introduced by numerical summation of series. Use of the binomial series especially for approximations.

(iv) *Logarithmic and Exponential Functions.* The graph ; change of scale of the graph ; change of base of logarithms ; theory and use of slide-rule. General theory of the functions (if calculus is available).

(v) The straight-line law and the law $y=ax^n$ deduced from a list of observed values.

(vi) *Partial Fractions*, if required for Calculus.

(vii) *Miscellaneous Graphs.* A discussion of the sign of the quadratic function ; graphs of quadratic functions and of other families of graphs, e.g. the graphs of functions of the form

$$(ax^2+2bx+c)/(Ax^2+2Bx+C)$$

s suggested as a stimulating subject.

(viii) *Successive approximation*, e.g. to the small value of x given by

$$4x^3-7x^2+2x=\tfrac{1}{30},$$

or to each of the roots of $(x-4)(x-2)(2x-1)=\cdot 1$.

(ix) *Horner's Method* of solving algebraic equations.

(x) *Nomography* ; representation of variables on parallel scales ; concurrence given by $v=au+b$; multiplication and division by the use of logarithmic scales.

(xi) *Annuities and Insurance.*

(xii) *Choice and Chance or Probability.*

(xiii) *Relativity.*

(xiv) *History of Mathematics.*

These suggestions are not intended to be exclusive or final. There are many pleasant byways in algebra. For example, no mention has been made of Theory of Numbers. Some care, however, is needed in the selection of work likely to prove suitable for non-specialist courses.

APPENDIX A 1

THE TEACHING OF DIRECTED NUMBER

[This section should be compared and contrasted with Appendix A 2, which presents views sharply opposed to those here presented.]

An essential feature of algebra is the extension of the number concept which it brings, and there is some difference of opinion as to the best method of treatment.

There are those who in effect say, 'Let us follow the historical development'; the others retort, 'No! science has reached a clear explanation of the whole matter: that is henceforward the proper starting point'.

This appendix is an attempt to state in general outline the case of those who are content, or who definitely prefer, to follow the historical development.

Historically the development of the number concept has been a rather accidental process not deliberately pursued but imposing itself on mathematicians sometimes imperceptibly, sometimes even against their will.

The original numbers were those of counting, i.e. the integers—absolute, cardinal, signless. Whether zero is a number in this sense may be a matter of dispute; certainly $\frac{1}{2}$ is not.

The first extension—both historically and in the experience of the modern child—is to fractions. Here it is worth noting how naturally the step comes about: it is all but, if not quite, effected in the change from 'three years and a half' to 'three and a half years'. In this connection the word 'times' plays a prominent part: 'he ran round the track three and a half times' is an advance on 'he ran round three times and then ran half-way round', but 'three and a half times' is still an expression where the original meaning of half (not yet a number) is possible: but in 'on the average he was late three and a half times a week' we have definitely crossed the Rubicon: so also in 'his average for the season was 27·35'.

If we ask ourselves what these new numbers 'mean', the answer is not easy; indeed, in any full sense there is no answer. What we know about them is the way they work—the rules according to which we combine them with one another and with the natural numbers. Beyond this the best we can do, at all events on the elementary plane of thought, is to put them in an ordered series with the natural numbers according to their relative magnitude.

But for the actual purposes of arithmetic we really define them by the rules according to which they work : ‘a thing *is* what it *does*’.

The practice of measurement and of subdividing the unit leads naturally to *concrete* fractions ($\frac{3}{4}$ of a ton, etc.), and men were practically forced to make room for *abstract* fractions in their arithmetic, though, as just stated, an explanation of what an abstract fraction *is* in itself is not forthcoming, nor for the purposes of arithmetic is one necessary.

The next step—the inclusion of surds—came about in a somewhat different way. The Greeks constructed a theory of proportion before they knew of the existence of such things—a theory still surviving in the 7th book of Euclid. The discovery that the side and diagonal of a square were incommensurable forced them to revise their principles, and the 5th book of Euclid contains the result. The result, however, appears more clearly for our present purpose in the work of Archimedes. In commencing his evaluation of π he begins with the assertion that $\sqrt{3}$ *lies between* $\frac{1351}{780}$ and $\frac{265}{153}$, and his final conclusion is that π *lies between* $3\frac{1}{7}$ and $3\frac{10}{71}$. Here the sense of a number as given by its position in the ordered series is explicit.

In a boy's experience to-day the historical process so far sketched corresponds to the growth that goes on in his mind, particularly in his exercises in measurement and in plotting graphs—he gets by degrees a sense of the relative positions of the various fractions—they come to have a value-meaning and cease to be mere formal expressions.

Now let us see how algebra—that is, calculation by letters—supplied and still supplies the urge to the invention of negative numbers. Take what is perhaps a side issue first as an illustration of the way we are affected by the working of the machine.

When we write down a series of algebraic terms $a+b-c$ ..., the first term has no sign in front of it, and it would be barefaced sophistication to put one there. But when we multiply $x+3$ by $x+2$ we get $x^2+3x+2x+6$, and the next step is to collect the terms in x: that is, we isolate in thought the partial expression $+3x+2x$, and before we know what we are doing we slip into accepting the presence of a sign before the first term. And we may even proceed to give thanks for the discovery that we can insert a sign without harm, for we can then state the law as to the order of terms in addition without any troublesome qualification.

In arithmetic no such qualification was necessary. ‘In adding a column of figures we may take them in any order’; but at the outset of algebra we cannot say ‘the terms of an algebraic expression may be taken in any order’ without specifying what is to be done with the signless first term.

Very largely because of this rule as to the order of terms we come almost unconsciously to regard the sign as belonging to the term, since it moves with it; and also we cease to object to the initial

term of an expression having a sign prefixed, even if that sign is a minus. So there is no longer anything shocking to us in such a statement as $-3x+2x=-x$, though we are still far from recognising the existence of negative numbers, and if we want to translate the statement into English we shall probably do so in terms of a calculus of operations and say 'the effect of first subtracting $3x$ and then adding $2x$ is the same as that of subtracting x'. None the less we have started on the path which will lead us to the explicit recognition of negatives.

But a further impulse arises if we arrange a piece of multiplication, not in a single line as above, but in rows as we do in arithmetic. Here we say, for instance: write down the product of $x-2$ by x and underneath it the product of $x-2$ by 3 *and add*

$$\begin{array}{r} x-2 \\ \underline{x+3} \\ x^2-2x \\ \underline{3x-6} \end{array}$$

The prescription 'and add' is a further urge towards absorbing the sign before the $2x$ in the term and regarding the addition as that of $3x$ to $(-2x)$.

In a subtraction sum this is still more manifest:

$$3x-(x+6)=3x-x-6=2x-6$$

raises no question; but if we write in rows

$$\begin{array}{r} 3x \\ \underline{x+6} \end{array}$$

with the mental direction 'subtract', we are faced directly with the problem of subtracting 6 from zero: in arithmetic we should get over the difficulty by borrowing: here we can only indicate that the subtraction has still to be performed by writing $2x-6$. None the less we are a step further towards regarding $0-6$ as an operation with an answer -6, whatever that may mean. If now or after further experience of a similar kind we make the plunge and decide to admit -6 as a formal answer to this queer sum, we gradually recognise that the minus sign has now a new function. In $0-6$ it indicated a verb in the imperative mood; in the answer -6 it indicates an adjective qualifying the 6, and it is advisable to read it as 'negative' and even to differentiate it in writing from the original verbal sign: this can be done either by writing (-6) or $\bar{6}$.

Two questions arise: (1) how do we handle these new expressions? (2) can we attach any meaning to them if they occur at stages of the work where meaning matters?

The answer to the first question is easy: these expressions are a mere convenience suggested to us, indeed imposed upon us by the working of the machine. As we don't know what the expressions

mean we cannot *prove* anything about them; we have simply to accept for them the rules of the machine. These rules as applied to the new expressions prove to be, 'To add a negative number means to subtract the corresponding arithmetical number', and so on.

In using negative numbers in this way we are not open to the reproach of using meaningless or undefined symbols. It is true that we cannot give a meaning to the negative number in itself, just as we could not give a meaning to the abstract fraction in itself—or, to go outside mathematics for an illustration, as we cannot give a meaning to a comma or semicolon in itself—still less to the blank left by the printer between words; all we know, and all we need to know, is how the symbol works while it is still in its place in the machine; or to speak in terms of definition, the negative number, like the abstract fraction, is defined by stating its properties: once more 'a thing *is* what it does'.

At this stage algebra stopped for a time: there was no harm in negative numbers as they turned up in the process of the work—they were a mere mechanical convenience; but if they occurred as a final result, in the solution of an actual problem, not of a mere formal equation, they indicated that something was wrong for they were meaningless; either the problem was impossible or there had been some misconception or wrong assumption in expressing it in algebraic form.

But it gradually appeared that in some cases, not in all, an intelligible result was suggested by a negative solution. To find the centre of parallel forces, we assume perhaps that it is *x to the right* of the mid-point of the bar on which the forces act; if x comes out negative, the primitive method would be to start again with the assumption *x to the left* and get a satisfactory result. But then comparing the two pieces of work we find that they are identical throughout except for the sign of x; so we might have said at once on getting the solution $x=\bar{3}$: the point required is 3 to the left. In a book of such comparatively recent date as Todhunter's *Algebra* there is an important chapter on the examination of such apparently meaningless, because negative, solutions of problems.

Thus the primitive algebra has for its subject-matter the signless numbers of arithmetic, supplemented by symbols adopted merely for convenience; these symbols are or may be meaningless in themselves, but that does not matter—all that is requisite is to know how they work. Trouble arises when attempts are made to 'prove' the rules as applied to these symbols—for instance, that $\bar{a}\times\bar{b}=ab$. As the symbols are in themselves meaningless nothing can be proved about them: the only place for logic in the matter is to show that the various rules adopted are mutually consistent and therefore permissible; subject to this the rules are arbitrary and are dictated by convenience.

All the same it appears in the long run that some sort of meaning

can be attached to the new symbols, analogous to that attached to fractions. A specialist in the Theory of Number does not regard a fraction as a number—to him it is a 'number pair'—but the ordinary citizen and the boy learning arithmetic or algebra do so regard it. They give it a sort of meaning by placing it in the ordered series. Just so a sort of meaning can be attached to negative numbers by regarding them as preceding zero. This raises an important question: can we conceive numbers preceding zero in the sense corresponding to that in which we regard numbers proper as succeeding zero? And the answer depends on what we mean by zero in a particular case. Zero may be absolute or relative; numbers consequently may also be absolute or relative. If we count the number of people in a room, or the number of feet in the distance between two points, there is an absolute zero and the number is absolute: in these cases the negative number has a formal existence only and is in itself meaningless. But where we name the date 1932, or the temperature 50° F., we are counting from an arbitrary point, and could count backwards and continue beyond that point: in these cases zero is arbitrary and the number is relative, and the negative number has a meaning of its own: the date $\overline{1000}$ means 1000 B.C.

These considerations do not affect the algebra itself; they only concern the interpretations we can put on the results of the algebra. If a problem leads to a negative number for a population or for the excess of a father's age over that of his son, it is not the algebra that is at fault—the algebra has exposed the fact that the problem proposed was impossible. But if a problem as to the price of eggs leads to a negative answer, it means that instead of the purchaser paying the vendor as we normally expect, the vendor had to pay the purchaser for taking the eggs away.

When we have arrived at these notions of an arbitrary zero and relative numbers we can complete our nomenclature by calling the original numbers positive in contrast to the new ones, to which alone we have hitherto attached an adjective. So we have, not two algebras, but a single algebra applicable to two possibly different fields of subject-matter, viz.:

(i) to the absolute signless numbers of arithmetic with the supplementary negative numbers—the latter a mere convenience, meaningless in themselves;

(ii) to the 'qualified' or 'relative' numbers, positive or negative.

The reference of these adjectives should be noted: 'qualified' merely expresses the fact that the numbers have an adjective attached; 'relative' that they are relative to an arbitrary zero. Up to this point there lies hidden a limitation which it is necessary to expose. A population is essentially an absolute number; change of population is relative, the expression $p+x$, where p is absolute

and x is relative, is legitimate, but x, if negative, must not be numerically greater than p if the expression is to be intelligible—just as in the particular case it must not be fractional. This does not affect the working of the machine: the statements

$$1000 + x = 600, \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$x = \overline{400} \quad \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

are formally identical and it does not matter whether we stop the working of the machine at (a) or at (b) with the comment 'this is impossible'.

Similarly in finding the centre of mass we frequently use a relative mass to cover the two cases of a piece added and of a piece taken away—but if the piece to be taken away proved to be greater than the whole, the problem would be proved 'impossible'. The limitation, it will be noticed, is not in the algebra itself, but in the interpretation.

The stage we have reached so far is exemplified early in the plotting of graphs: no one has difficulty in the notion of extending the time line backwards and taking negative numbers to indicate dates before the epoch or in measuring ordinates downwards to represent, say, an adverse balance at the bank. And in fact the notion of *relative* number, like that of fractional number, has passed into popular use. The difficulties, as with fractions, become more serious when we consider multiplication.

Multiplication by a fraction does require a new or extended definition of multiplication, and this fact is exemplified in the frequent perplexity felt at the phenomenon of 'multiplication' making a number smaller. Just so does multiplication by a negative number require a new definition—and even when the definition is learnt, time and growth are necessary before the new idea is really appropriated.

Our starting point as always is the working of the machine: we are not concerned to create new concepts and then to devise an algebra to express them, but to take the existing algebra and to see if we can find useful applications for those of its products which are on the face of them meaningless.

Now the general rule for multiplication of two expressions is: multiply each term of the one by each term of the other and put the resulting partial products together with a certain rule of signs, viz.: for instance, $(a-b)\times(x-y)\equiv ax-bx-ay+by$. This rule of course is based and proved on the supposition that $a>b$ and $x>y$. We cannot use it to prove that $(-b)\times(-y)=+by$. What we can do is to say that if we want to use the rules without reference to the limiting conditions subject to which they were established we must make a new definition, viz. that $(-b)\times(-y)=+by$ and $a\times(-y)=-ay$.

This, it may be noticed again, is precisely what we have already done in regard to fractions. $7 \times \frac{3}{5}$ is *a priori* meaningless and nothing can be *proved* about it; but the ordinary rule of multiplication is that $a \times b = b \times a$, therefore if we want to retain the rule we must take or define $7 \times \frac{3}{5}$ as equivalent to $\frac{3}{5} \times 7$, which is intelligible.

All we have done so far is to adopt a new definition forced upon us by the working of the machine. It remains to be seen whether the new definition will serve any useful purpose. This involves the examination of such questions as are considered in the main report (pp. 56-58), and it turns out that the definition proves in fact to be exceedingly convenient.

The crown of the historical development is reached only when we include the so-called 'imaginary' quantities. The machine invented them—forced them on men's attention very much against their will. They were necessary for the perfect working of the machine and they could be used to get many beautiful formal results—but had they any meaning in themselves and could they be trusted unless results suggested by them could be checked independently? In the end these questions were satisfactorily answered, but the opinion may be hazarded that, if the machine had not created the expression $x+iy$, Argand would not have invented his diagram which gives it a meaning.

It is important to realise that with this last extension—the inclusion of $\sqrt{(-1)}$—the field of ordinary algebra is complete. The working of the machine evolves no fresh types, and though further types of numbers can be and have been invented, e.g. by extending Argand's diagram to three dimensions, they are found to require new algebras differing in their rules from the primitive algebra with which we are concerned.

The spatial representation of complex number in Argand's diagram has reacted on the representation of positive and negative number, and it is more usual now to call these 'directed' rather than 'relative'. This sharpens the distinction between the original absolute signless numbers and the generalised numbers of algebra: if we call p absolute and x *relative*, $p+x$ is a natural and proper expression, but if we call x *directed*, with a clear notion in our minds of all that 'directed' implies, we cannot add x to p. Such considerations do not affect the working of the machine, and any attempt to use its rules to prove, e.g., that an absolute number 'is equal to' a directed number is of course merely ridiculous: the machine does not distinguish, and *so far as it is concerned* there is no difference between 2 and positive 2: this no more means that 2 and positive 2 are identical than it means that x and 2 are identical.

APPENDIX A 2

GENERALISATIONS OF NUMBER. DIRECTED NUMBERS

[Those who hold the views expressed in this appendix differ from the point of view of Appendix A 1. A consequence, as regards teaching, is that they believe in a somewhat later and more deliberate introduction to directed numbers.]

(·1) We shall have occasion to refer, in this appendix, to various kinds of numbers and it will be convenient to adopt a definite nomenclature throughout.

The *natural numbers*, 1, 2, 3, 4, 5 ..., will be taken for granted. A discussion of these numbers, such as may be found in *Principia Mathematica*, belongs to the region of mathematical logic which separates the territory of mathematics from that of philosophy. We shall not invade that territory, but it may be mentioned here that, from this logical point of view, the natural numbers are 0, 1, 2, 3, ... and not merely 1, 2, 3, ...

The other kind of numbers which arise in elementary arithmetic, numbers like

$$\tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{5}{1},\ \tfrac{7}{2}\ \ldots,$$

we call *fractions*.

The term signless may be applied to the natural numbers or to the fractions.

Two kinds of *directed numbers* may be introduced: the directed integers $(+1,\ +2,\ +3,\ -1,\ -2,\ \ldots\ 0)$ and the directed numbers in the more general sense

$$(+\tfrac{2}{3},\ -\tfrac{2}{3},\ +\tfrac{1}{8},\ -\tfrac{4}{1},\ -\tfrac{7}{2},\ 0,\ \tfrac{5}{2}\ \ldots),$$

but this distinction need not be emphasised here. In mathematics directed numbers are more often called 'rational', but the term 'directed' seems more appropriate for the kind of introduction that has to be given to the learner. Directed numbers are subdivided into positive numbers, the number zero, and negative numbers.

Real numbers are those which are usually defined by the method of Dedekind, modified by later writers, and these are subdivided into *rational-real* and *irrational-real* numbers. Irrational-real numbers are often called, shortly, irrationals, and the rational-real numbers have in the past often been confused with the directed numbers [see section (·3) below]. In this appendix we shall not use

the words 'rational' and 'irrational' except hyphened with 'real' in the sense just explained.

Numbers like $\sqrt{(-1)}$, $-i$, $a+bi$ we call *complex*.

(·2) The extension of the term 'numbers' to denote 'fractions' was the first step in a series of generalisations, and this step was taken in very early times. There followed an interval of some centuries in which signless numbers were the only ones that entered into consideration. For many purposes the signless numbers are sufficient; if there are some sheep in a field the number of them must be one of the natural numbers; and if the price of an article is x shillings, x may be one of the fractions. The laws of algebra were developed and the machinery was set in motion on a signless basis. Sometimes it was found that a meaningless result was reached or that some operation was indicated by the symbols but could not be carried out. At first this position was accepted; thus Diophantus, having obtained the equation $4x+20=4$, remarks 'which is absurd'.

The various extensions of the term number were made, most of them at a much later date, in order to make possible the inverse arithmetical operations of subtraction, division, and involution, or from the algebraic standpoint, to render certain equations soluble which were previously considered, as in the example from Diophantus, absurd. In one way this procedure was illogical. It is not a correct principle that what is true sometimes is true always; $x=a-b$ follows from $x+b=a$ in signless algebra provided that $a > b$, but the correct conclusion from $x+5=3$ in signless algebra is that no such number as x exists. And there is no conclusive reason why the inverse operations should always be possible. But it gradually came to be perceived that, if the laws of algebra (as derived from the arithmetic of signless numbers) were taken to be binding on certain meaningless symbols, many useful results could be obtained. These results could often be verified by other methods; e.g. on p. 63 of this report an example in which a result is found by using a negative number can be seen to be true independently of that idea, and, as another example, if the factors of x^4+4 are found by using complex numbers, it is easy to verify them by writing the expression in the form $(x^2+2)^2-(2x)^2$.

Geometrical and other illustrations of the meaningless symbols were also found which increased confidence in their use. Thus in spite of a certain hesitancy, which is shown by the use of the words surd, absurd, irrational and imaginary, new kinds of numbers came to be employed many years or even centuries before their theory was completely understood. For example, Euler, 200 years ago, used $\sqrt{(-1)}$ and obtained many useful results with it, although he knew as well as any modern schoolboy that a negative number can have no square root.

It was only in the latter part of the 19th century that mathematicians began to clear up their ideas on the subject of number. 'Up to the middle of the 19th century', one writer * has remarked, 'no subject was less logical than mathematics'.

It is now realised that it is possible, starting from the natural numbers, to deduce the whole theory of the remaining species of numbers. The principle of the method of deduction which was introduced by Weierstrass and developed by more modern writers may be described in the following way. The idea of 'number' is formed by a series of generalisations starting from natural numbers; the laws concerning the four fundamental operations as applied to natural numbers are the subject of the early chapters of arithmetic. Fractions, which can be considered as pairs of natural numbers, are then introduced, and the definitions of equality and of the four fundamental operations have to be stated anew for these numbers; these would be:

$$\frac{a}{b} \text{ and } \frac{c}{d} \text{ are called equal if } ad=bc,$$

$$\frac{ad+bc}{bd} \text{ is called their sum and is denoted by } \frac{a}{b}+\frac{c}{d},$$

etc., and the ordinary arithmetical rules can then be shown to hold good for fractions.

In this method a fraction is a pair of natural numbers, and that is just what a fraction ought to be. To put the matter crudely, if we are measuring a distance in inches and find that it is less than an inch, we use some other unit, say a 'line' (i.e. an old unit, 12 of which go to the inch), and find perhaps that the distance is 5 lines. The distance is then determined by the pair of numbers 5 and 12; of these the 12 tells us the new unit, and the 5 is the number of these units in the distance. The real advance is made when we begin to regard the pair of numbers as a single number $\frac{5}{12}$ and say that the distance is $\frac{5}{12}$ inches. A fraction is then a 'number' in a new sense of that word, and it is just because the sense *is* new that definitions of 'sum', 'product', etc., are needed.

To go deeper into the meaning of a/b would be to return to the domain of mathematical logic (cf. Whitehead and Russell, *Principia Mathematica*).

The next step in the process of generalisation comes in the early part of algebra when the new idea of directed numbers is introduced; here again the meanings of equality, sum, product, etc., as applied to the new numbers, have to be defined. Then, logically, the next step is the verification that these numbers obey the laws of algebra. The actual procedure of teaching is different and is discussed in section (·5).

* P. E. B. Jourdain, *The Nature of Mathematics* (Jack: London).

Real numbers are the next to be introduced, and this is the step which presents the greatest difficulty. For practical purposes the matter is not one of great urgency in teaching; although there is no number p/q whose square is 2 the schoolboy uses the symbol $\sqrt{2}$ to denote a number, it may be $\frac{14}{10}$, $\frac{141}{100}$, $\frac{1414}{1000}$, whose square is, for the purpose in hand, near enough to being 2.

The theory of complex numbers, on the other hand, presents little difficulty to the ordinary sixth-form boy. The geometrical interpretation of complex numbers was discovered early in the 19th century by Wessel, Argand, and Gauss, but its importance was not for some time sufficiently appreciated by others. It was only in the early years of the present century that writers of text-books and their critics began to call attention to the insufficiency of the common procedure. 'Nothing can be more mistaken', says Whitehead,* 'than to suppose that meaningless symbols can be made to yield valid proofs of propositions. If they are used, every result obtained by means of them is obtained subject to the proviso: provided that a valid interpretation of the symbols exists; a symbol that has not been defined is a mere blot of ink on the paper, and nothing is proved by a succession of blots except the existence of a bad pen or a careless writer.' The matter was also made very clear, as regards complex numbers, in G. H. Hardy's *Pure Mathematics*, which appeared in 1907.

Under the influence of such writers the reform of the teaching of complex numbers should have been universal. The corresponding change for directed numbers is not such a simple matter because it concerns pupils of 11 or 12 instead of pupils of 15 or 16. The principles involved are the same in both cases, and therefore the only excuse for adhering to the old-fashioned method for directed numbers would seem to be that the newer method is too difficult to teach. It is clearly impossible that a complete logical theory of directed numbers should be presented to the schoolboy, but it is suggested that there are certain fundamental matters which ought to be treated with proper regard to that theory. [See section (·5).] The position is the same as for other difficult parts of mathematics: the teacher is to omit or postpone everything which is too difficult or too academic for his pupils, but he ought to abstain from filling the gaps with misleading theories.

(·3) The teacher should understand the principle according to which the mathematician makes use of the same symbol with different meanings in different places. The fraction $\frac{2}{1}$ is in fact denoted by the same symbol as the natural number 2, and this symbol serves also for the directed number $+2$ since the $+$ is often omitted; later the same symbol is used for the real number 2 and even for the complex number $2+0i$.

* *Introduction to Mathematics* (Home Univ. Libr.).

This device of notation is a most important one. There is a (1, 1) correspondence between the fractions of type $x/1$ and the natural numbers x; corresponding to any property of the natural numbers there is a property of these fractions which, when the device is adopted, will be expressed by precisely the same symbols; when the laws of operation as applied to fractions have been proved, it will be easy to see that the abbreviation of $x/1$ into x must always give the correct results. Similarly the natural numbers have a (1, 1) correspondence with the positive directed integers, and the rational-real numbers with the directed numbers. Using the same symbol for the different kinds of numbers is a most important economy in the work of learning the technique.

In spite of the theoretical objection to using the same symbol with different meanings, it is found that there is not much confusion caused by it in practice. For example, if the fractions are being arranged 'in order', a meaning is required for $a/b < c/d$. The definition of equality gives $a/b = ad/bd$ and $c/d = bc/bd$, and this suggests that the definition of 'less than' should be

$$a/b < c/d \text{ is defined to hold if } ad < bc;$$

it follows from this definition that, say, $\frac{2}{1} < \frac{11}{4} < \frac{3}{1}$, and although this is written in the abbreviated form $2 < 2\frac{3}{4} < 3$, it does not really mean that the fraction lies between two natural numbers. There is no need to consider or even define inequalities like $n < c/d$ where n is a natural number; yet the mathematician is able with impunity to write down such inequalities without pausing to reflect on the fact that the n is standing as an abbreviation for $n/1$.

But, in spite of the convenient symbolism, it is sometimes important to bear in mind the distinctions between different kinds of numbers; fractions with denominator 1 are *not* the same as natural numbers, positive integers are *not* the same as signless integers, rational-real numbers are not the same as directed numbers, and complex numbers are never under any circumstances real. All such suppositions are erroneous 'and must be discarded if correct definitions are to be given'. This distinction was emphasised by Russell as long ago as 1903 in his *Principles of Mathematics*: 'when mathematicians have effected a generalisation of number, they are apt to be unduly modest about it'.

There is, in fact, not one algebra but many algebras; there is the algebra (i) of natural numbers, (ii) of fractions, (iii) of directed numbers, (iv) of real numbers, (v) of complex numbers. There are also other algebras, such as the vector algebra, in which $a \times b$ has a different meaning from $a \,.\, b$ and in which $b \times a$ is not the same as $a \times b$, but in this report we are chiefly concerned with (i) to (v). Some theorems are identical in form in the different algebras; others are true in one and false in another. Thus if a, b, c have the appro-

priate meanings, '$c(a+b)=ca+cb$' is true in any of the algebras just mentioned, 'x exists such that $x^2=2$' is false in (i), (ii), (iii), but true in (iv), and the equation $x^4=1$ has *one* root in (i) or (ii) *two* roots in (iii) or (iv), and *four* roots in (v). We may say that each algebra has its own set of counters and each has its own rules. There is a close similarity between the rules of different algebras, but the algebras are introduced as different games, not as different parts of the same game. Each algebra has its own domain of application; some algebras are suitable for certain purposes, others for other purposes.

We only use one kind of algebra at a time, and the question of which it is does not often arise.

(·4) The boy learns in elementary algebra that the square of a number is positive and that $\sqrt{-1}$ does not exist; when he begins algebra (v), he begins to use the symbol -1 with a new meaning, and the fact that he can now interpret $\sqrt{-1}$ does not mean that he has found a square root of a negative number; he should understand that he is working in a new algebra and that the -1 is now complex, not negative.

Again, it is not legitimate to argue that $x+5=3$ *must* have a root and that therefore negative numbers must exist, or that $x^2=-1$ must have a root and that therefore complex numbers must exist. We do not argue that we must introduce the number ∞ because the equation $1/x=0$ must have a root; this argument was indeed not altogether unknown in England at the end of the last century, but it would not at any rate have been claimed even then that the new number (∞) must obey all the laws of algebra; there was, however, a tendency to make this claim for negative numbers on similar grounds. The fact that we want $x+5=3$ or $x^2+1=0$ to be soluble is an argument for creating a new algebra, not for altering the elementary algebra which is perfectly correct without any alteration.

Now that clear ideas have been reached on these subjects, is it necessary to perplex our pupils of to-day with the same vagueness and obscurity through which the early pioneers had to struggle? * If the answer is 'No', what kind of a treatment of directed numbers is to be presented to the schoolboy? Formal proofs of the laws of operations and the determination of a minimum set of laws from which the others can be deduced is wholly unsuitable work for schools; only the exceptional sixth-form boy will be interested in it.

Some care should be exercised about the kind of examples presented to the student before he has reached directed numbers. This is thought by many teachers to be a tiresome restriction, but it is

* E. V. Huntington in Young's *Monographs on Modern Mathematics* (Longmans).

possible to follow de Morgan's advice: * 'If the student should meet with an equation in which positive and negative numbers stand by themselves, such as $(+ab) \times (-c) = -abc$, let him for the present reject the example in which it occurs and defer the consideration of such equations'. If an equation like $10 - x = 7$ occurs while the boy is in the stage of signless algebra (and if the answer is not guessed), the equation should be written $10 - 7 = x$ (not $-x = 7 - 10$);

$$\therefore\ 3 = x; \quad \therefore\ x = 3.$$

But it may occur to the boy to strike out the 10's from the equation $10 - x = 10 - 3$ on the ground that equal numbers may be subtracted from each side. Some teachers would say that this is a point at which directed numbers ought to be introduced. Those who wish to postpone the introduction would merely remind the boy that he cannot subtract 10 from 7.

For the details of the way in which it is here suggested that directed numbers ought to be introduced we must leave the teacher to consult a good modern text-book, but it will be worth while, in sections (·5) and (·7), to emphasise certain points of importance which have sometimes been neglected or misunderstood.

(·5) The new numbers must be introduced in a practical example in which there is an obvious convenience in them; the example is not to be an artificial one constructed for the apparent purpose of introducing a new game with symbols.

One convenient starting point is furnished by a thermometer graduated above and below some point of reference (A), such as the freezing point of water. When $+1$, $+2$, $+3$, ... are used as reference marks for the points above A, and -1, -2, -3, ... for the points below A, the signs $+$ and $-$ are used in a sense that will be new to the pupil; in the original sense in which he will have met them before (in arithmetic and in the early part of algebra) they express commands to add or subtract. To distinguish the new use, it would be desirable to modify the signs, or to write them above the numbers, but the usual device is to write directed numbers as

$$(+1),\ (+2),\ (-1),\ \ldots,$$

which is convenient for printing. The object of the notation is to avoid confusion between the command 'subtract 3' and the directed number (-3).

It will be understood that the thermometer is to be only one of several illustrations that are used in introducing the subject. See page 56.

When addition and subtraction have been defined informally, and it should be clear from section (·3) that they need definition, expressions like $(+2) - (+3)$ and $(+2) + (-3)$ will occur in which the signs inside and outside the brackets have two different meanings.

* *The Study and Difficulties of Mathematics* (Open Court: Chicago).

By means of practical illustrations, such as movements up and down a ladder graduated like the thermometer, pupils should be led to the so-called 'rules of signs' according to which, for example, $-(+N)$ is written $-N$.

Illustrations should also be given to show the convenience of definitions like $(+a)+(+b)=(+a+b)$.

(·6) It is sometimes supposed that, once addition has been defined, multiplication can be deduced. In one sense this is true: repeated addition gives a meaning to the multiplication of x by a natural number y, not only when x is a natural number, but even when it is a fraction or a sum of money, or a weight, or a directed number. Thus $x \times 3$ is taken to mean $x+x+x$ and is sometimes denoted by $3x$, and $(+5)\times 3=(+5)+(+5)+(+5)=(+15)$.* But this is 'one-way' multiplication only and it assigns no necessary meaning to $3\times(+5)$. Nevertheless $(+5)\times 3=(+15)$ can be used to suggest that the appropriate meaning to be given to $(+5)\times(+3)$ is $(+15)$; it does not *prove* that this definition *must* be given, and there is in fact no need to introduce $(+5)\times 3$ at all. What is wanted is definitions of $(+a)\times(+b)$, $(+a)\times(-b)$, $(-a)\times(+b)$, and $(-a)\times(-b)$. There can be no question of proving these results; for example, it is quite illogical to deduce the last one from the formula

$$(x-a)(y-b)=xy-ay+ab-xb \ (x>a,\ y>b)$$

of signless algebra by the substitution $x=y=0$, but this formula, and others like $s=ut$, can be used to show what a nuisance it would be to have any different definitions.

If the distinction between the two kinds of multiplication is ignored, certain fallacies may arise. For examples see Appendix A 3. In a similar way if it was thought necessary in dealing with complex numbers to introduce both $(a+bi)\times n$ and $(a+bi)\times(n+0i)$, they would both be defined as $na+nbi$, and it might be concluded (fallaciously) that $n=n+0i$.

(·7) A considerable course of drill examples is needed to impress upon the pupil that the machinery of directed number algebra works in just the same way as the machinery of arithmetic. This takes the place of proofs of formal laws of operation which would only be appreciated by older pupils. In the course of this work the 'bracket' notation for directed numbers should gradually be dropped so that the pupil begins to use the same notation as well as the same machinery in the two kinds of algebra.

The teacher will understand that $(+2)-(+3)$ and $(+2)+(-3)$ correspond to fundamentally different operations although they are to be denoted, except in the elementary stage, by the same symbol $2-3$; this is part of the economy of notation explained in section (·3).

* × means 'multiplied by' not 'times.'

Some examples of 'simple equations' and of 'problems' which involve directed numbers should be included in the course.

(·8) The graphs of simple functions can be used to illustrate operations with directed numbers, but the teacher should be clear in his own mind about what is being illustrated by the graphical work. It can be shown, for example, that the definitions already given for directed numbers will cause the points of the graph of $4-2x$ to lie on a straight line; we assume that $4-2x$ is an expression of 'directed algebra', i.e. that the 4 of $4-2x$ means $(+4)$, and either the 2 means $(+2)$ or else $2x$ is being used as an abbreviation for $x+x$, and also x is a directed number; if we put, say, $(+3)$ for x we get $4-2x=(+4)-(+3)-(+3)=(-2)$, not $(+4)+(-2)(+3)$. It is not legitimate to argue that the collinearity of points on the graph amounts to a proof of the rules for directed numbers, since it is not known independently of these formulae that the graph need be a straight line: indeed, apart from the formulae, the graph only exists for $0 < x < 2$.

(·9) It is interesting to note that the need for different kinds of algebra has its counterpart in analytical geometry. When the student begins that subject he is already acquainted with directed numbers, and it is not usually thought worth while to construct an analytical geometry of signless numbers; but there are very important distinctions to be drawn between different kinds of geometry, those in which coordinates are real and those in which they are complex, those in which there are points at infinity and those in which there are not. It is by distinguishing the various types of geometry that we can avoid the difficulties about imaginary points and points at infinity that used to perplex the student.* An essential preliminary is to be clear about the algebras.

Besides the books already quoted the teacher may with advantage consult:

J. W. Young's *Fundamental Concepts of Algebra.* (Macmillan.)

T. P. Nunn's *Teaching of Algebra.* (Longmans.)

C. V. Durell's *Teaching of Elementary Algebra.* (Bell.)

and for an easy account of the semi-philosophical part of the subject:

B. Russell's *Introduction to Mathematical Philosophy.* (Geo. Allen and Unwin.)

* See Hardy's *Pure Mathematics*, Appendix IV.

APPENDIX A 3

THE DISTINCTION BETWEEN SIGNLESS NUMBERS AND POSITIVE DIRECTED NUMBERS. SOME FALLACIES

[For reasons explained in Appendix A 2 this distinction is more important in theory than in practice, and the teacher will be well advised not to worry his pupils with any of the difficulties that are discussed in the present section. On the other hand, with mathematical specialists, some of the points may arise and will be worth clearing up.]

(·1) The importance of the distinction between signless and directed numbers is emphasised by those who hold the views expressed in Appendix A 2. The signless number 3 is quite different from the positive directed number $(+3)$. A man who takes 3 steps and then takes 2 steps has taken 5 steps altogether. One who takes 3 steps forward and then 2 steps forward is 5 steps away from the starting point. But if a man takes 3 steps and then takes 2 steps forward it is impossible to know his distance from the starting point without knowing the direction of the first 3 steps.

(·2) Consider the following statements which are commonly made in elementary algebra:

$$\text{(i) If } x \neq 0,\ \frac{ax}{x} = a.$$

$$\text{(ii) If } x \neq 0 \text{ and } ax = bx, \text{ then } a = b.$$

$$\text{(iii) } ax + bx = (a+b)\,x.$$

Let x be a positive directed number, say $(+2)$, and let a be the signless number 3. Then $ax = 3(+2)$, which is an abbreviation for $(+2)+(+2)+(+2)$ and therefore equals $(+6)$; hence $\frac{ax}{x} = \frac{(+6)}{(+2)}$, which is $(+3)$, and so, if (i) is assumed, we get $(+3) = 3$, a directed number equal to a signless one, which is absurd.

Again, $\quad 3(+2) = (+6)$ and $(+6) = (+3)(+2)$

and so, if (ii) is assumed, we get $3 = (+3)$.

Finally, if $a = 3$, $b = (+5)$, and $x = (+2)$,

$$\begin{aligned} ax + bx &= 3(+2) + (+5)(+2) \\ &= (+6) + (+10) = (+16), \end{aligned}$$

whereas $(a+b)\,x$ contains the meaningless factor $3 + (+5)$.

(·3) The fallacy is most transparent in (ii); the equality of $3(\ +)$ and $(+3)\times(+2)$ was obtained by defining them both as $(+6)$, and thereby two different sorts of multiplication have been defined, namely, the product of two directed numbers and the product of a directed number by signless 3 in the ordinary sense of repeated addition. Corresponding to these two kinds of multiplication there are two kinds of division and the conclusion $3=(+3)$ is only reached by applying two different processes, viz. the two kinds of division, to the two sides of the equality.

In (i) $\dfrac{ax}{x}$ means $(a\times x)\div x$, but the sense in which $\div$ is used is not the reverse of the sense in which $\times$ is used; thus there is no reason why the result should be a.

(·4) In pointing out that the division of ax by x when a is signless is a different operation from the division of bx by x when b is directed, and that therefore '$a=b$' is not always a consequence of '$ax=bx$', we appear to be destroying the simplicity of ordinary algebra. We are at least making it apparent that there are difficulties in the way of working with two kinds of numbers at once.

Expressions like $3(x)$, where (x) is a directed number, and $3(+2)+(+5)(+2)$, where 3 is a signless number, ought not to occur in directed algebra. If $3(x)$ *does* occur it is a mere abbreviation for $(x)+(x)+(x)$, and the expression that ought to occur instead of $3(+2)+(+5)(+2)$ is

$$(+2)+(+2)+(+2)+(+5)(+2);$$

if this is to be factorised, it should be factorised as

$$(+2)\,[(+1)+(+1)+(+1)+(+5)]$$

and not as $\quad (+2)\,[1+1+1+(+5)]$.

In directed algebra $\quad (+2)\div(+2)=(+1)$, not 1.

(·5) Again, consider a question about population. The population of a town is p; it increases by x or perhaps it decreases by x, and so it becomes $p+x$ or $p-x$. Instead of being content to leave it at that we may (and often do) wish to take (x) as a directed number and take the new population to be $p+(x)$ whether (x) is positive or negative.

In doing this we are electing to use directed numbers. We ought therefore to use them throughout, and to represent the original population by a directed number (p).

Is the population, then, a directed number or a natural number? The answer is that the population can be represented by either kind of number.

In a problem for which signless numbers are sufficient it is always *possible* to use directed numbers instead. This is because of the

exact correspondence which exists between the signless numbers and the positive directed ones (and also between the various operations thereon). In making a list of populations there would be no point in using directed numbers, but if changes of population are in question, then it is useful to work in directed numbers.

By keeping to one kind of number at a time we preserve the simplicity of our algebra and avoid such difficulties as those mentioned on p. 103. The statements (i), (ii), (iii) on that page are valid when a, b, x are all signless and also when they are all directed. The fallacies arise out of a hybrid algebra.

APPENDIX A 4

FACTORISATION OF TRINOMIALS

Factors of the following type are academic exercises rather than exercises of much practical value, but for those who wish to use them the following hints are offered.

(·1) In the case of trinomials in which the coefficients of the 1st and 3rd terms may be factorised in many ways, the number of trials may be restricted by the application of the following principles :

Taking ax^2+bx+c as the typical trinomial :

(i) if a number is a factor of b and its square is a factor of a or c, it must be a factor of a term in each bracket ;

(ii) if a or c is a multiple of a square and b is prime to it, the square will not be distributed between the two brackets.

Consider $8x^2-15x-27$:

Applying (i), 27 must factorise as 3×9.
" (ii), 8 " " " 1×8.

And the only trials necessary are

$$(8x\quad 3)(x\quad 9) \text{ and } (8x\quad 9)(x\quad 3).$$

Again, these principles determine that $24x^2-3x-196$ cannot be factorised, 3 being a factor of a and b but not of c; the expression might have factorised if a had been a multiple of 3^2, not otherwise.

(iii) In such a case as $8x^2-83x-720$, a and c are multiples of powers of 2, and b is prime to 2 ; $\therefore$ 8 factorises as 8×1 and 720 as 16×45 ; 48×15 ; 80×9 ; 144×5 ; 240×3 ; 720×1. But principle (ii) vetoes 48×15 and 240×3, because each of the factors is a multiple of 3 whereas 3 is not.

Again, since 2 is not a factor of 83, no even factor of 720 can go with the same bracket as the 8 (the coefficient of x^2), so that the only trials necessary are :

$$(8x\quad 1)\ (x\quad 720),$$
$$(8x\quad 5)\ (x\quad 144),$$
$$(8x\quad 9)\ (x\quad 80),$$
$$(8x\quad 45)\ (x\quad 16),$$

and the last, which is correct, will probably be tried first.

Unusually complicated illustrations have been selected here ; the same ideas apply to easier cases.

(·2) It is often helpful to group terms by degree,

e.g. $$a^2-a-b^2+b$$

may be grouped as $$(a^2-b^2)-(a-b),$$

but it may also be grouped as a trinomial function of a, giving

$$a^2-a+b(1-b),$$

$$(a \quad b)\{a \quad (1-b)\} \text{ for trial.}$$

Again, $$x^2-y^2-x-7y-12$$

may be grouped as a trinomial function of x,

$$x^2-x-(y^2+7y+12).$$

But it may be grouped by degree as $(x^2-y^2)-(x+7y)-12$ for factorisation by the trinomial method :

$$\{(x+y) \quad 4\}\{(x-y) \quad 3\}.$$

APPENDIX A 5

DEVELOPMENT OF ALGEBRAIC FORM

Development of the sense of form and the power to recognise identity of form under variety of appearance is a very important part of training in algebra. It generally comes in a rather haphazard fashion and is apt largely to be missed except by those to whom the subject is naturally congenial. It is possible that if the point were more definitely recognised and more deliberate attention given to it, more widespread success might be attained.

The more complicated cases of factorisation afford an illustration. The formula $a^2-b^2 \equiv (a-b)(a+b)$ and its simplest applications, such as $73^2-57^2=16\times 130$, present little difficulty to anybody. The usual practice is to follow these with such expressions as $(x+y)^2-a^2$ *presented as examples to be factorised.*

It is suggested that the following alternative procedure might be found useful:

'In the formula $A^2-B^2\equiv(A-B)(A+B)$, what do A and B represent?' 'Numbers.' 'Yes; either actual numbers, such as 7 and 5, or ... ?' 'Letters representing numbers.' 'Yes; x and y of course, but is $2x+3$ also a number?' 'Yes.' 'Then may we put $2x+3$ for A and 5 for B?' 'Yes.' 'Then please do so, i.e. copy the formula carefully, putting $(2x+3)$ for A and 5 for B.' So we get $(2x+3)^2-5^2\equiv\{(2x+3)-5\}\times\{(2x+3)+5\}$, and this can be reduced to $(2x-2)\times(2x+8)$. 'Now, is this correct? Please verify it by expanding each side separately.'

Other formulae, e.g. $(A+B)^2\equiv A^2+2AB+B^2$, should be treated in the same way—the object being not so much to teach factorisation as to develop the power of extending formulae from their simple to more complicated shapes; which will of course incidentally, but very effectively, develop the power of recognising the fundamental form under the more complicated appearance. See also § 5·1.

APPENDIX A 6

THE SUMMATION OF A GEOMETRIC PROGRESSION

The summation of a G.P. may be approached inductively as follows: tabulating for the series $1+2+4+\ldots$

U_n	1	2	4	8	16	32	64
S_n	1	3	7	15	31	63	127

it will be noticed that $S_n = U_{n+1} - 1$.

If a similar tabulation for $1+3+9+\ldots$ is made, it will be seen that $S_n = \frac{1}{2}(U_{n+1} - 1)$, for $1+4+9+\ldots$ that $S_n = \frac{1}{3}(U_{n+1} - 1)$, and the general result for $1+r+r^2+\ldots$ will appear as $S_n = \dfrac{U_{n+1} - 1}{r-1}$; and this will suggest the formal method of proof for the general formula.

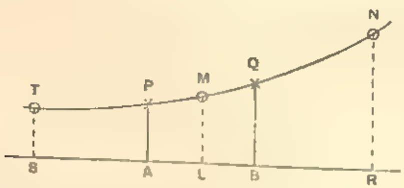
APPENDIX A 7

THE GRAPH OF a^x

The study of the graph of a^x for different values of a throws much light on the theory of logarithms in general and on the question of the change of the base of the logarithms in particular. The first stage is to sketch rapidly 2^x and 3^x,* taking about 5 points on each, and to notice their general properties, viz. the value is always positive and rises steadily as x increases; the rise is slow for x-negative, from zero to 1 when x is 0, afterwards it is very quick up to a when x is 1 and thence to ∞.

For a careful drawing it is best to take a only a little more than 1, e.g. to consider $1{\cdot}1^x$ and to point out how it can be used (i) as a table of compound interest at 10%, (ii) as a table of logarithms to base 1·1.

Next it may be well to illustrate the simplicity of the construction of the curve, thus:



If the x-axis AB and any two points on the curve, P above A and Q above B, are given, any number of other points may be found.

Thus M corresponding to L, the mid-point of AB, is given by

$$ML^2 = PA \,.\, QB.$$

" N " " " R when $BR = AB$ is given by

$$NR = QB^2 \div AP.$$

" T " " " S when $SA = AB$ is given by

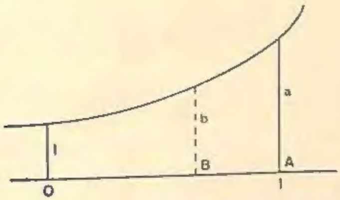
$$TS = PA^2 \div QB,$$

and so on for as many points as may be required.

* At a later stage, one inserts between these the graph selected so that the gradient is unity when x is zero, which is the graph of e^x.

Thus any of the curves is the locus of a point which moves so that to abscissae in A.P. correspond ordinates in G.P., and two points of the curve (together with the x-axis) suffice to determine the curve.

Finally, if any one of the curves is given, and also the line of the x-axis,



let us label with 0 the point A when the ordinate is 1,
and with 1 ,, ,, ,, ,, ,, a.

The curve with this scale on the x-axis is the graph of a^x, and the abscissae are the logarithms of the corresponding ordinates to base a.

If, however, OB is taken as unity instead of OA and this new scale marked on the axis, the curve automatically becomes the graph of b^x, and the abscissae are the logarithms of the corresponding ordinates to base b. For example OA is $\log_b a$ with OB as 1. But by this change the number giving any abscissa will have been multiplied by OA/OB or divided by OB/OA, that is, have been multiplied by $\log_b a$ or divided by $\log_a b$. Thus to change logarithms from base a to base b we merely multiply by the constant OA/OB, which is $\log_b a$, or divide by the constant OB/OA, which is $\log_a b$.

APPENDIX A 8

A NOTE ON CORRECTING EXERCISES

In manipulative work in Algebra it is sometimes possible for the pupil to obtain a correct result as a consequence of two balancing errors or by a fundamentally wrong method. Hence in correcting such work it is necessary to look through the steps of the working and not merely the answer.

Some examples illustrating these points follow :

(i) $a^2 - ac - b^2 + bc$

$$= a(a-c) - b(b-c)$$
$$= (a-b)(a+b-c).$$

$a+b-c$ is derived from $a-c$ and $b-c$, the $-c$ being common.

(ii) $c^2 - 2bc - a^2 + b^2$

$$= c(c-2b) - (a^2 - b^2)$$
$$= c(c-2b) - (a-b)^2$$
$$= [c + (a-b)]\,[(c-2b) - (a-b)]$$
$$= (c-b+a)(c-b-a).$$

(iii) $\dfrac{x^2 - a^2}{x-a} = x + a$, by ' cancelling ' x into x^2 and a into a^2.

(iv) $\dfrac{a^3(b-c) + b^3(c-a) + c^3(a-b)}{a^2(b-c) + b^2(c-a) + c^2(a-b)} = a + b + c$, by 'cancelling' $(b-c)$, etc.

(v) $\dfrac{x}{3} - \dfrac{x-4}{4} = \dfrac{4x}{12} - \dfrac{3(x-4)}{12} = \dfrac{4x}{12} - \dfrac{3x+12}{12} = \dfrac{4x - 3x + 12}{12}.$

(vi) $\dfrac{2^3 . 4^3 . 16^{-n}}{16^2 . 4^{-2n}} = \dfrac{22^{11-n}}{20^{2-2n}} = \dfrac{22^{11-n}}{20^n} = 22^{11-n} - 20^n = 2^{11}.$

(vii) There is a disadvantage in setting questions to which right answers may be obtained by working containing the commonest mistakes boys make. (See *Math. Gaz.* vol. xi No. 160, p. 177.)

Solve : $\dfrac{bx}{a} - \dfrac{d}{c} = \dfrac{a}{b} - \dfrac{cx}{d}.$

'Clear of fractions', says the boy, and multiplies one side by ac and the other by bd;

$\therefore\ bcx - ad = ad - bcx$;

$\therefore\ x = \frac{ad}{bc}$. This answer is right.

Simplify $\frac{y-z}{yz} + \frac{z-x}{zx} + \frac{x-y}{xy}$.

'Take the L.C.M.', that is, multiply the whole expression by zyx, making it $x(y-z) + y(z-x) + z(x-y)$, which equals 0, the right answer.

Other types to avoid concern circles with radius 2 units (when $\pi R^2 = 2\pi R$), journeys made at two speeds divided equally as regards time (giving the right average speed by an unjustifiable assumption), and all questions in which a factor which boys are apt to neglect is equal to unity.

APPENDIX A 9

A MULTIPLICATION TABLE FOR DIRECTED NUMBERS.

The following multiplication table presents the rules for multiplying directed numbers independently of any mode of interpretation. It appeals to those capable of appreciating the abstract side of the subject.

Starting with

	-3	-2	-1	0	1	2	3
3				0	3	6	9
2				0	2	4	6
1				0	1	2	3
0							
-1							
-2							
-3							

One fills up blanks by taking numbers in regular order from the sequence $-9, -8, \ldots -1, 0, 1, 2, \ldots 9 \ldots$ and gets first

	-3	-2	-1	0	1	2	3
3	-9	-6	-3	0	3	6	9
2	-6	-4	-2	0	2	4	6
1	-3	-2	-1	0	1	2	3
0					0	0	0
-1					-1	-2	-3
-2					-2	-4	-6
-3					-3	-6	-9

One then fills in the remaining blanks, either by continuing the left-hand column downwards, or by continuing the bottom rows backwards.